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Paper Discussions

ONE of the most important reasons for presenting a paper at a technical convention is that interested members of the audience can discuss the subject of the paper with the author. Although these discussions may benefit the entire audience and add considerably to the value of the paper, it is unfortunate that discussions do not always evolve because the paper may not be well understood in a short presentation time or because the interested members of the audience may not have time to formulate their question sensibly. Consequently, with a lack of discussion, many well organized sessions are not as valuable as they should be.

This problem has been alleviated somewhat at the WESCON Convention and other technical meetings on the West Coast. Here fewer papers are presented in each session, but a panel is selected to discuss the papers after they have been presented. Audience participation is encouraged, and the panel may include the authors themselves in a round-table discussion after all the papers have been given. Since the panel is given some previous knowledge of the subjects to be presented, a discussion does occur, and, subsequently, members of the audience contribute their thoughts which add considerably to the value of the paper presentations.

There are other methods of obtaining discussions of papers, and one of them is to obtain written discussions which are published in an appropriate journal. This is a particularly significant method of obtaining discussion of TRANSACTIONS ON AUTOMATIC CONTROL papers because, although the papers receive the benefit of reviewer's comments, they are not always presented at a conference, and they are not usually exposed to open discussion. We are fortunate to have a few discussions of papers in this issue. We believe that the most valuable discussions are made voluntarily, and we would like to receive more discussions for publication if they add to the clarity or broaden the scope of any paper either directly or by comparison with an alternate method.

The only other requirement, which may lead to some editorial censorship, is that the discussions by both the discussor and the authors must be written in a courteous manner. Of course, it is not always easy for an author to reply politely to a discussion which appears to be inappropriately or unjustly made. However, this could be a fault in the organization and clarity of the paper itself, not obvious to the author, which could have misled many readers, and any further clarification, however unnecessary it may seem to the author, could add to the value and prestige of the paper.

We hope that the few discussions in this issue will be found interesting, and that they will evoke more discussions in future issues. Our correspondence section will be open for notes, sometimes brief papers, and discussions, as well as announcements and all informal documents not subject to review. Actually the correspondence is published as rapidly as possible without the review delay, because we believe that more of this informal correspondence coupled with the larger, formal papers will add considerably to the interest and value of the TRANSACTIONS.

—The Editor

The Issue in Brief

Stability of Systems with Randomly Time-Varying Parameters, A. R. Bergen

Due to intermittent failures, a system may alternate in a random fashion between different configurations having stable and unstable modes. The resulting stability of such a system is discussed in this paper. It is investigated conveniently by using matrix notation and Kronecker products. This paper is of particular interest because it is an excellent example of the powerful mathematical techniques that are becoming more widely used in the analysis and synthesis of control systems.

Dynamic Programming Approach to a Final-Value Control System with a Random Variable Having an Unknown Distribution Function, M. Aoki

A number of papers have been written in recent years about the application of dynamic programming to control systems. In this paper the technique is applied to a final-value control system. Given a criterion of performance, the deviation of the process, and the domain of the control variable, a sequence of control variables is determined as a function of the state vector of the system and time in a manner that optimizes system performance.

It is shown that the determination of the control variables must be made sequentially rather than specified as a function of the initial state and time because of the stochastic nature of the problem. By means of a functional equation technique of dynamic programming, a recurrence relation of the criterion function of the process is derived.

The concept of suboptimal policy is introduced and numerical experiments are performed to test suboptimal policy suggested from a numerical solution of the recurrence relationship.

Mathematical Aspects of the Synthesis of Linear Minimum Response-Time Controllers, E. B. Lee

The synthesis of minimum response time control systems has been of interest for many years. In this paper a procedure is presented for finding, systematically, the forcing function to be applied to the process to give time-optimal control. The results are limited to the case where the process to be controlled has dynamics which are completely known and are adequately described by a system of linear differential equations.

The implication of the method presented is that any process as limited above can be time-optimally controlled provided that a system of transcendental equations can be solved. The results have been used to indicate that the method can be applied in quite general situations.

Statistical Design of Digital Control Systems, J. T. Tou

Although the problem of statistical design of digital control systems has been considered in the literature, a simplified design procedure is introduced in this paper. The useful minimum mean-square error performance criterion is adopted as the performance criterion, and the system design is based upon the Wiener-Kolmogoroff theory of optimum filtering and prediction. The treatment of this optimization problem makes use of the modified z-transform technique, and it is presented in close parallel with the procedure for the statistical design of linear continuous-data control systems. By making use of some important properties of the auto-correlation sequence, the cross-correlation sequence, the pulse spectral density and the pulse cross-spectral density, it is shown that the statistical design of digital control systems can be carried out in a simple and systematic manner.

Determination of Periodic Modes in Relay Servomechanisms Employing Sampled Data, H. C. Torng and W. E. Meserve

Sampled-data feedback systems containing a relay and a zero-order hold have been extensively investigated. The describing function and the phase-plane approach has been used with some success in studying these systems, but these methods have been rather unwieldy.

In this paper, the difference equation approach is used to reveal all possible periodic modes of relay sampled-data feedback systems of any order in a simplified way. This is achieved by expanding periodic functions of an integral argument in terms of orthogonal functions in the discrete domain.

The effect of the presence of a dead zone in the relay characteristic is studied. The steady-state response of the system corresponding to step, ramp, or sinusoidal inputs is also discussed.

Analysis of Pulse Duration Sampled-Data Systems with Linear Elements, R. E. Andeen

Most of the extensive literature concerning sampled-data systems is based on a pulse amplitude modulation process. In this paper, the sampled data is represented by constant amplitude pulses, with the pulse width modulated by the data. The characteristics of these systems are described, and an analytical means for studying the performance of these systems is presented. Analytical results are compared with experimental tests, and a short discussion of the relative advantages and disadvantages of pulse duration modulation in control system is included.

The Analysis of Cross-Coupling Effects on the Stability of Two-Dimensional, Orthogonal, Feedback Control Systems, D. B. Newman

Multipole or multidimensional control servos have been receiving considerable attention in the past few years, and this paper should serve as an excellent introduction to the subject for those not acquainted with the problems involved and the methods of analysis. In addition to a development of an algebraic model of the system, matrix methods of analysis are introduced, and the performance of the system is interpreted in terms of the pole-zero plots of the system. A technique showing how an unstable system may be made stable by the introduction of appropriate cross-coupling is discussed.

Stabilization of Linear Multivariable Feedback Control Systems, E. V. Bohn

This paper is another in the field of multipole control systems. Like the previous paper, it may serve as an introduction to the subject although it is somewhat more general. This paper is also concerned with stabilization techniques for multivariable feedback control systems. A method of system stabilization is discussed which reduces the problem to essentially that of a single variable system for which stabilization techniques are well developed. This method uses a compensating matrix which is either the inverse system—or transposed system-matrix. An example of this method of system stabilization is given and its usefulness discussed.

Correspondence

A paper by G. J. Murphy and N. T. Bold[1] has resulted in an interesting discussion between J. B. Cruz and one of the authors, Murphy. A note on a particular nonlinear feedback system is provided by B. Chatterjee, and Z. Bonenn presents an illuminating comparison between various methods of determining stability including the describing function and Lyapunov's second method. The root locus may be modified to factor nth order polynomials as described by J. D. Glomb, and methods of determining the transfer function coefficients of a linear dynamic system from frequency response characteristics are discussed by P. V. Rao and E. L. Bai.

PGAC Membership Directory

This list of control engineers is again published to denote the active members in the field.

[1] G. J. Murphy and N. T. Bold, "Optimization based on a square-error criterion with an arbitrary weighting function," IRE Trans. on Antennas and Propagation, vol. AC-5, pp. 24-30; Jan., 1960.

Stability of Systems with Randomly Time-Varying Parameters*

A. R. BERGEN†, MEMBER, IRE

Summary—**The stability of a system which alternates between stable and unstable configurations at random times may be investigated conveniently using Kronecker products. The system is stable with probability one if, and only if, all the eigenvalues of a specified matrix lie within the unit circle. If stability in the presence of a parameter adjustment is to be investigated, a root-locus plot of the eigenvalues is convenient.**

I. Introduction

A CONSIDERATION of systems with randomly time-varying parameters by Rosenbloom[1] in 1954, and more recently in two papers by Bertram and Sarachik[2] and Samuels[3] indicates a growing interest in such systems. This paper is concerned with random systems of a particular type, those in which the system alternates between two known configurations at random times. The alternation between configurations is a purely deterministic process; the system remains in each configuration for a random interval. It is assumed that the random intervals are mutually independent random variables.

One may presume such systems to arise as a result of an intermittent faulting of the system in some way, by an intermittent component failure or by a random interruption of a control or guidance loop. In either case it is assumed that the system adopts one of two possible configurations. For simplicity it is assumed also that the order of the system is the same in either configuration.

In a case of interest to control system designers one of these two configurations is unstable. The question then arises as to the stability of the system when this unstable state is intermittent. Presumably the system will be stable if the unstable state is induced infrequently and is of short enough duration.

Since the system switches randomly between stable and unstable modes it is necessary to define precisely what is meant by stability. A time-invariant linear system, initially in equilibrium, is said to be stable (strictly stable) if, and only if, the system returns to equilibrium after any finite disturbance. In the case of the random system presently under discussion, however, the behavior following a disturbance can only be described probabilistically.

The return of the system to equilibrium is analogous to the convergence of a sequence of random variables, and, borrowing the terms of the statistician, the system may exhibit convergence in probability, convergence in the mean (norm), or convergence with probability one. Of these, convergence with probability one is the most desirable type of behavior assuring that for any bounded disturbance the system returns to equilibrium after sufficient time with probability one; this behavior matches closely that of stable time-invariant linear systems. For the purposes of this paper a stable system will be defined as one which returns to equilibrium with probability one after a bounded disturbance.

II. Analysis

The problem under consideration is the stability of the autonomous randomly time-varying linear system described in the kth interval by the vector differential equation,

$$\dot{x}(t) = A_k x(t), \quad t_{k-1} \leq t < t_k \qquad k = 1, 2, \cdots, \tag{1}$$

$x(t)$ is an n-vector called the state of the system with components $x_1(t), x_2(t), \cdots, x_n(t)$, called the state variables; for an nth order system there are n such state variables. In the kth interval, A_k is a constant $n \times n$ matrix.

The solution of (1) is[4]

$$x(t) = e^{A_k(t-t_{k-1})}x(t_{k-1}), \qquad t_{k-1} \leq t < t_k \qquad k = 1, 2, \cdots, \tag{2}$$

this being the solution in a particular (the kth) interval. In cases of practical interest the matrices A_k are bounded, and therefore the state variables are continuous functions of time. Then

$$x(t_k) = e^{A_k T_k}x(t_{k-1}) \qquad k = 1, 2, \cdots, \tag{3}$$

where $T_k = t_k - t_{k-1}$ is the kth interval.

Eq. (3) relates the system state at the end of the kth interval to that at the end of the $k-1$st interval and, by iteration, to the initial state $x(t_0)$. The stability of the system depends on the behavior of $x(t_k)$ after a large number of random transformations, the typical one given by (3). It should be recalled that the randomness of the system resides in the intervals T_k; the A_k are a

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[1] A. Rosenbloom, "Analysis of Linear Systems with Randomly Time-Varying Parameters," *Proc. Symp. on Information Networks*, Polytechnic Inst. of Brooklyn, Brooklyn, N. Y., vol. 3, pp. 144–154; April, 1954.

[2] J. E. Bertram and P. E. Sarachik, "Stability of circuits with randomly time-varying parameters," IRE Trans. on Circuit Theory, vol. CT-6, pp. 260–270; May, 1959.

[3] J. C. Samuels, "On the mean square stability of random linear systems," IRE Trans. on Circuit Theory, vol. CT-6, pp. 248–259; May, 1959.

[4] B. Friedman, "Principles and Techniques of Applied Mathematics," John Wiley and Sons, Inc., New York, N. Y., pp. 122–124; 1956.

purely deterministic sequence of alternatively stable and unstable matrices.[5]

Eq. (3) is convenient for investigating the limiting mean of $x(t_k)$, and this will be done to illustrate the technique to be used later when stability is considered. Iterating (3),

$$x(t_k) = e^{A_kT_k}e^{A_{k-1}T_{k-1}} \cdots e^{A_kT_k}x(t_0), \tag{4}$$

and taking expectations (the intervals are assumed independent),

$$E\{x(t_k)\} = E\{e^{A_kT_k}\}E\{e^{A_{k-1}T_{k-1}}\} \cdots E\{e^{A_1T_1}\}x(t_0). \tag{5}$$

$x(t_0)$ may be assumed to be an arbitrary nonrandom disturbance. Assuming now that the alternate intervals are identically distributed random variables, alternate terms of (5) will be identical, and for k even,

$$\begin{aligned} E\{x(t_k)\} &= [E\{e^{A_2T_2}\}E\{e^{A_1T_1}\}]^{k/2}x(t_0) \\ &= \Psi_A{}^{k/2}x(t_0), \end{aligned} \tag{6}$$

where Ψ_A is an abbreviation for the bracketed expression. For k odd (6) is premultiplied by $E\{e^{A_1T_1}\}$; since this single transformation cannot affect the limiting behavior of $x(t_k)$, it is sufficient to consider (6) with k even.

From (6), the mean of $x(t_k)$ tends to zero with k if, and only if, the eigenvalues of Ψ_A lie within the unit circle. This condition is analogous to the condition imposed upon the poles of a stable sampled-data system, *i.e.*, the poles must lie within the unit circle.

The problem of evaluating Ψ_A may be simplified considerably if the eigenvalues of A_1 and A_2 are distinct. In this case, if the integral converges,[6]

$$E\{e^{A_kT_k}\} = \int_0^\infty e^{A_kT_k}p_k(T_k)dT_k = M_k(-A_k) \qquad k = 1, 2. \tag{7}$$

Here $p_k(T_k)$ is the probability density function of stable (unstable) intervals. $M_k(-A_k)$ is a matrix obtained in the following way:[7] find $M_k(s)$, the Laplace transform of $p_k(T_k)$, and replace s by $-A_k$ and 1 by the identity matrix. Note that it is not necessary for the distribution of intervals to be the same in the stable and unstable cases.

The convergence of the mean of $x(t_k)$ with k does not imply stability since it is entirely possible for the mean to converge while the variance diverges. It is therefore necessary to investigate the behavior of a norm of $x(t_k)$. It will be shown, however, that it is possible to investigate the behavior of a norm in a manner entirely similar to that used for the mean. This may be accomplished by the use of Kronecker or direct products of matrices.

Kronecker products are related to tensor products; for a treatment of the subject the reader is referred to MacDuffee,[8] or Bellman.[9] Bellman[10,11] has used the Kronecker product in problems related to that treated here. Kronecker products of matrices are defined in Appendix I. An important property of such products needed in the derivation which follows is also given in Appendix I.

Consider the behavior of the vector $y(t)$, which is the Kronecker product of the state vector $x(t)$ with itself.

$$y(t) = x(t) \otimes x(t). \tag{8}$$

The cross symbol denotes the Kronecker product. Consider $\dot{y}(t)$ in a particular interval:

$$\begin{aligned} \dot{y}(t) &= \dot{x}(t) \otimes x(t) + x(t) \otimes \dot{x}(t) \\ &= A_kx(t) \otimes x(t) + x(t) \otimes A_kx(t) \\ &= [(A_k \otimes I) + (I \otimes A_k)]y(t). \end{aligned} \tag{9}$$

The last line in (9) follows by premultiplying $x(t)$ (in the second line) by the identity matrix, using (15) of Appendix I, and (8). Under consideration then is a new system (with state vector y) which has been derived from the old system (with state vector x). The mean of the new vector y may now be investigated in the same way in which the mean of the x vector was investigated. But since the vector y includes among its components the squares of the components of the x vector, it is possible to investigate the mean-square behavior of the state variables.

The use of (9) without modification leads to unnecessary computational labor. The vector y contains among its elements components such as x_1x_2 and also x_2x_1; these elements are identical and it is possible, by eliminating such redundant elements, to reduce the order of the matrix equation from dimension n^2 to dimension $n(n+1)/2$. How this reduction proceeds may be illustrated as follows. Suppose the same element appears in the second and fifth rows of the y vector. Eliminate this element the second time it appears, *i.e.*, eliminate the fifth row. At the same time modify the matrix which premultiplies y by adding the fifth column to the second and then eliminating the fifth row and column. Continue until all the redundant elements are removed. Clearly, this process removes all the redundant elements while maintaining the correct relations between the remaining elements.

Let the reduced matrix and vector be designated by B_k and z respectively. Then (9) after reduction is

$$\dot{z}(t) = B_kz(t), \quad t_{k-1} \leq t < t_k \qquad k = 1, 2, \cdots \tag{10}$$

[5] Clearly the stability of the system does not depend on whether A_1 is stable or unstable. Assume then that it is stable.

[6] To test the convergence of the integral replace the matrix A_k by the real part of the eigenvalue with the largest real part. If the integral does not converge, the mean of x does not converge and it is therefore unnecessary to compute Ψ_A. In the case of the matrix A_1 with eigenvalues having negative or zero real parts, the integral always converges; the test is unnecessary.

[7] Friedman, *op. cit.*, pp. 113–117.

[8] C. C. MacDuffee, "The Theory of Matrices," Chelsea Publishing Co., New York, N. Y., pp. 81–86; 1956.

[9] R. Bellman, "Introduction to Matrix Analysis," McGraw-Hill Book Co., Inc., New York, N. Y.; 1960.

[10] R. Bellman, "Limit theorems for non-commutative operators, I," *Duke Math. J.*, vol. 21, pp. 491–500; September, 1954.

[11] R. Bellman, "Kronecker products and the second method of Lyapunov," *Math. Nachr.*, vol. 20, pp. 17–19; May, 1959.

with solution (at the switching times)

$$z(t_k) = e^{B_k T_k} z(t_{k-1}), \qquad k = 1, 2, \cdots. \tag{11}$$

Eqs. (11) and (3) are now in the same form. The method previously used to investigate the convergence of the mean of $x(t_k)$ can now be used to investigate the convergence of the mean of $z(t_k)$; in the derivation simply replace x by z and A by B. It follows that if the eigenvalues of B_1 and B_2 are distinct[12] and the integral corresponding to (7) converges, the behavior of the mean of $z(t_k)$ depends on the locations of the eigenvalues of

$$\Psi_B = M_2(-B_2)M_1(-B_1). \tag{12}$$

If these eigenvalues lie within the unit circle, the mean of z tends exponentially to zero and so do each of its components which include $E(x_1^2), E(x_2^2), \cdots, E(x_n^2)$. In this case the expected norm for the disturbed system tends to zero with k. Moreover, because of the exponential convergence it is possible to establish the stronger condition of convergence with probability one. This is shown in Appendix III.

Example

The stability of the feedback system shown in Fig. 1 is to be determined. The minor loop shown opens intermittently; the system is stable with the loop closed and unstable with the loop open. Assume that the loop is closed during the first interval.

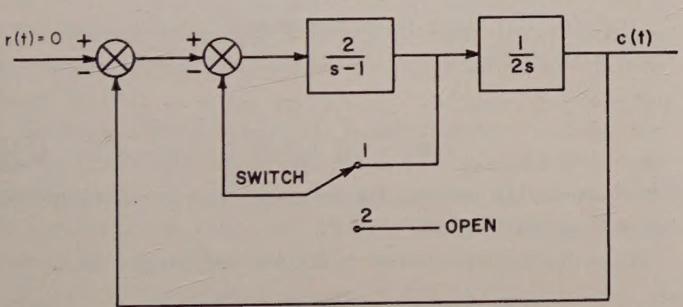

Fig. 1—Random system of example.

The scalar differential equation describing the autonomous system in the kth interval is

$$\ddot{c} + (-1)^{k-1}\dot{c} + c = 0.$$

Choose for the state variables, $x_1 = c$ and $x_2 = \dot{c}$. Then in the kth interval

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & (-1)^{k-1}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix},$$

and in the notation of (1),

$$A_k = \begin{bmatrix}0 & 1\\ -1 & (-1)^{k-1}\end{bmatrix}.$$

Consider next the matrix

$$A_k \otimes I + I \otimes A_k = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & (-1)^{k-1} & 0\\ 0 & -1 & 0 & (-1)^{k-1}\end{bmatrix} + \begin{bmatrix}0 & 1 & 0 & 0\\ -1 & (-1)^{k-1} & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & (-1)^{k-1}\end{bmatrix}$$

$$= \begin{bmatrix}0 & 1 & 1 & 0\\ -1 & (-1)^{k-1} & 0 & 1\\ -1 & 0 & (-1)^{k-1} & 1\\ 0 & -1 & -1 & 2(-1)^{k-1}\end{bmatrix}$$

which is found by using (13) in Appendix I. The y vector corresponding to x is

$$y = \{x_1^2, x_1x_2, x_2x_1, x_2^2\}.$$

Here the second and third rows are seen to be equal. The third row may be eliminated, yielding the z vector

$$z = \{x_1^2, x_1x_2, x_2^2\}$$

with the corresponding B_k matrix

$$B_k = \begin{bmatrix}0 & 2 & 0\\ -1 & (-1)^{k-1} & 1\\ 0 & -2 & 2(-1)^{k-1}\end{bmatrix}.$$

The B_k matrix is formed from the preceding matrix of Kronecker products by adding the third column to the second and then eliminating the third row and column.

In particular then

$$B_1 = \begin{bmatrix}0 & 2 & 0\\ -1 & 1 & 1\\ 0 & -2 & 2\end{bmatrix}$$

$$B_2 = \begin{bmatrix}0 & 2 & 0\\ -1 & -1 & 1\\ 0 & -2 & -2\end{bmatrix}.$$

Consider now the eigenvalues of B_1 and B_2 respectively. These can be computed directly or, as shown in Appendix II, they can be specified in terms of the eigenvalues of the lower order matrices A_1 and A_2 respectively. The eigenvalues of A_1 are

$$-\frac{1}{2} + j\frac{\sqrt{3}}{2} \quad \text{and} \quad -\frac{1}{2} - j\frac{\sqrt{3}}{2};$$

those of B_1 are therefore $-1+j\sqrt{3}$, -1, $-1-j\sqrt{3}$.

[12] In Appendix II, it is shown that while the eigenvalues of the matrix involving the Kronecker products [shown in (9)] are not distinct, those of the reduced matrix B_k ordinarily are if the original matrix A_k has distinct eigenvalues. In any case the eigenvalues of the reduced matrix B_k may be enumerated using the rule of Appendix II.

Similarly, the eigenvalues of A_2 are

$$\frac{1}{2}+j\frac{\sqrt{3}}{2} \quad \text{and} \quad \frac{1}{2}-j\frac{\sqrt{3}}{2};$$

those of B_2 are therefore $1+j\sqrt{3}$, 1, $1-j\sqrt{3}$. Since the eigenvalues of B_1 and B_2 are distinct, (12) may be used to investigate stability.

First, however, a model for the random opening of the minor loop must be chosen.

Assume, for example, as a plausible model, a purely random opening and closing of the minor feedback loop. Let the average interval during which the loop is closed (stable) be fixed at 1 second while the average interval during which the loop is open (unstable) is adjustable; let this average interval be $\overline{T}_2$.

In either case, the probability density function of intervals is exponential

$$\begin{aligned} p_k(T_k) &= \frac{1}{\overline{T}_k} e^{-T_k/\overline{T}_k} & 0 \le T_k \\ &= 0 & T_k < 0, \end{aligned}$$

with Laplace transform

$$M_k(s) = (\overline{T}_k s + 1)^{-1}.$$

$\overline{T}_k$ is the average interval.

If $\overline{T}_2$ is sufficiently small the system presumably is stable, while if $\overline{T}_2$ is large enough, system instability is to be expected. System stability as a function of $\overline{T}_2$ will now be investigated.

Note first that the defining integral for $M_2(-B_2)$ does not converge for $\overline{T}_2$ greater than unity. Then for $\overline{T}_2$ greater than unity the system is unstable. For $\overline{T}_2$ less than unity the eigenvalues of Ψ_B as given by (12) must be investigated. For the present model

$$\begin{aligned} \Psi_B &= (-\overline{T}_1 B_1 + I)^{-1}(-B_1 + I)^{-1} \\ &= \frac{1}{\overline{T}_2}\left[(B_1 - I)\left(B_2 - \frac{1}{\overline{T}_2} I\right)\right]^{-1}. \end{aligned}$$

Equivalently, and more conveniently, the eigenvalues of Ψ_B^{-1} may be examined. In Fig. 2 are shown the root-loci of the eigenvalues of Ψ_B as $\overline{T}_2$ is varied from zero to one. For $\overline{T}_2$ less than approximately $\frac{1}{3}$, the system is stable. For $\overline{T}_2$ greater than approximately $\frac{1}{3}$, one of the eigenvalues lies outside of the unit circle and the system is unstable.

Conclusion

In general it is difficult to obtain strong conditions defining the stability of systems with randomly time-varying parameters. In the particular case considered here it is possible to obtain sharp stability conditions, *i.e.*, the system is stable with probability one if, and only if, all the eigenvalues of a specified matrix lie within the unit circle. The system for which this applies is one which alternates between stable and unstable configurations. The times of switching from one state to the other are random.

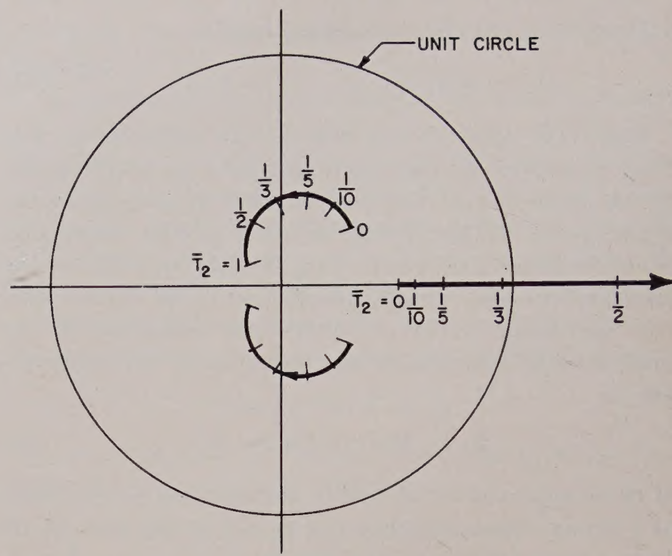

Fig. 2—Root-loci of eigenvalues.

Appendix I

The Kronecker product of two $n\times n$ matrices A and B (denoted by $A\otimes B$) is an $n^2\times n^2$ matrix defined as follows:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{n1}B & & \cdots & a_{nn}B \end{bmatrix}. \tag{13}$$

a_{ij} are the elements of A.

In particular the Kronecker product of the state vector x with itself is the n^2 dimensional vector

$$\begin{aligned} x \otimes x = \{&x_1^2, x_1x_2, \cdots, x_1x_n, x_2x_1, x_2^2, \cdots, \\ &x_2x_n, \cdots, x_nx_1, x_nx_1, \cdots, x_n^2\}, \end{aligned} \tag{14}$$

which includes among its components the squares of the components of x.

An important property of Kronecker products is

$$(AB) \otimes (CD) = (A \otimes C)(B \otimes D). \tag{15}$$

Appendix II

Under consideration are the eigenvalues of the reduced matrix B_k. Assume that the eigenvalues of the original $n\times n$ matrix A_k are distinct. Then

$$A_k\xi_i = \mu_i\xi_i \tag{16}$$

where the ξ_i are the n linearly independent eigenvectors corresponding to the distinct eigenvalues μ_i. Consider the right hand Kronecker product of (16) by $I\xi_j$. Using (15),

$$(A_k \otimes I)(\xi_i \otimes \xi_j) = \mu_i(\xi_i \otimes \xi_j). \tag{17}$$

Consider also the left hand Kronecker product of

$$A_k\xi_j = \mu_j\xi_j \tag{18}$$

by $I\xi_i$

$$(I \otimes A_k)(\xi_i \otimes \xi_j) = \mu_j(\xi_i \otimes \xi_j). \tag{19}$$

Adding (17) and (19),

$$[A_k \otimes I + I \otimes A_k](\xi_i \otimes \xi_j) = (\mu_i + \mu_j)(\xi_i \otimes \xi_j). \tag{20}$$

The matrix on the left of (20) is just the matrix in (9), *i.e.*, the B_k matrix before contraction. The eigenvectors of this matrix may be identified as the n^2 Kronecker products of the n eigenvectors of the matrix A_k having for eigenvalues the n^2 possible sums of eigenvalues of the original A_k matrix. Included among these sums are terms such as $\mu_1+\mu_2$ and $\mu_2+\mu_1$; hence the eigenvalues are not distinct.

Consider now the eigenvalues of the reduced matrix B_k. First note from (20) that

$$[A_k \otimes I + I \otimes A_k][(\xi_i \otimes \xi_j) + (\xi_j \otimes \xi_i)] = (\mu_i + \mu_j)[(\xi_i \otimes \xi_j) + (\xi_j \otimes \xi_i)]. \tag{21}$$

Now $\xi_i \otimes \xi_j + \xi_j \otimes \xi_i$ is a vector which displays the same structure regarding redundancies as does $x \otimes x$. This can be shown by writing this vector as a sum of self Kronecker products.

This being the case, a contraction of the type discussed in the text of this paper eliminates $n(n-1)/2$ of the redundant elements leaving the relation between the remaining elements intact; *i.e.*, the vectors which are the reduced versions of $\xi_i \otimes \xi_j + \xi_j \otimes \xi_i$ are the eigenvectors and the $\mu_i+\mu_j$ are the corresponding eigenvalues of the matrix B_k. But now, it is to be noted, the interchange of indices does not yield a new eigenvector; the $n(n-1)/2$ eigenvectors which were generated by such interchanges have been eliminated. The eigenvalues of B_k, therefore, are all possible $\mu_i+\mu_j$, with i less than or equal to j.

While the eigenvalues of A_k are distinct, it does not follow that those of B_k are distinct, although this will usually be the case. For example, if the eigenvalues of A_k are -1, -2, and -3, those of B_k will be -2, -3, -4, -4, -5, -6.

Appendix III

Assume the eigenvalues of Ψ_B lie within the unit circle. Then the infinite sum of vectors $E\{z(t_k)\}$ converges:

$$\sum_{k=1}^{\infty} E\{z(t_k)\} < \infty. \tag{22}$$

The necessary and sufficient condition for this convergence is that $\Psi_B{}^k$ tend to zero with k;[13] this is assured if the eigenvalues of Ψ_B lie within the unit circle. Convergence of the sum is also assured for each component of $E\{z(t_k)\}$. In particular

$$\sum_{k=1}^{\infty} E\{x_i{}^2(t_k)\} < \infty \qquad i = 1, 2, \cdots, n. \tag{23}$$

But now, by Tchebycheff's inequality,

$$\sum_{k=1}^{\infty} P[\,|\,x_i(t_k)\,| > \epsilon] < \frac{1}{\epsilon^2} \sum_{k=1}^{\infty} E\{x_i{}^2(t_k)\} < \infty,$$

where ϵ is an arbitrary positive number. Then by the Borel-Cantelli lemma[14] $x_i(t_k)$ can exceed the bound ϵ only a finite number of times in an infinite run. Since this holds for arbitrarily small ϵ, $x_i(t_k)$ tends to zero with probability one. Since this is true of all the components of $x(t_k)$, $x(t_k)$ itself tends to zero with probability one; the system is stable.

Acknowledgment

The author wishes to thank D. Slepian of the Bell Telephone Laboratories for pointing out some very useful properties of Kronecker products.

[13] V. N. Faddeeva, "Computational Methods of Linear Algebra," Dover Publications, Inc., New York, N. Y., pp. 61–62; 1959.

[14] W. Feller, "Probability Theory and Its Applications," John Wiley and Sons, Inc., New York, N. Y., pp. 154–155; 1950.

Dynamic Programming Approach to a Final-Value Control System with a Random Variable Having an Unknown Distribution Function*

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Summary—As described in the introduction of this paper, the state vector x_n, representing the present state of a sampling control system, is assumed to satisfy a difference equation

$$x_{n+1} = T(x_n, r_n, v_n),$$

where v_n is the control vector of the system subject to random disturbances r_n. The random variables r_n are assumed to be independent of each other and to be defined in a parameter space in the following manner: Nature is assumed to be in one of a finite number, q, of possible states and each state has its own parameter value, thus specifying uniquely and unequivocally the distribution function of r_n. It is further assumed that we are given the *a priori* probability $z=(z_1, \cdots, z_q)$ of each possible state of nature,

$$i = 1, 2, \cdots, q, \sum_{i=1}^{q} z_i = 1, \qquad z_i \geq 0.$$

Given a criterion of performance, the duration of the process and the domain of the control variable, a sequence of control variables $\{v_n\}$ is to be determined as a function of the state vector of the system and time, so as to optimize the performance.

The sequential nature of the determination of v_n is evident here, since the stochastic nature of the problem prevents specification of such a sequence of v's as a function of the initial state and time. By means of the functional equation technique of dynamic programming, a recurrence relation of the criterion function of the process, $k_n(x, z)$, is derived, where x is the state variable of the system when there remain n control stages.

When q is taken to be two and the criterion of performance to be x_N^2 with the constraint on the control variables

$$\sum_{i=0}^{N-1} v_i^2 \leq K,$$

where x_N is the final state vector of the system, it is shown that $k_n(x, z)$ is the form $w_n(z)x^2$ and the optimal $v_n(x, z)$ is linear in x, as might be expected. The dependence of $k_n(x, z)$ and $v_n(x, z)$ on z is investigated further, and $w_n(z)$ is found to be concave in z. Explicit quadratic forms for $w_n(z)$ are obtained in the neighborhood of $z=0$ and 1. The optimal $v_n(x, z)$ is found to be monotonically decreasing in z.

When the domain of v_n is restricted to a finite set of values, as in contactor servo systems, no explicit expressions for $k_n(x, z)$ and $v_n(x, z)$ are available. However, $k_n(x, z)$ is still concave in z and approximately given by

$$zk_n(x, 1) + (1-z)k_n(x, 0).$$

By solving the recurrence relation numerically, this approximation is found to be very good for moderately large n, say 10. This means that if one has explicit solutions for $p=p_1$ and $p=p_2$, then one can approximate the criterion function for the adaptive case where $Pr(p=p_1)=z$ by a linear function in z as shown above.

This provides fairly good lower bound on $k_n(x, z)$, and in turn serves to determine an initial approximate policy.

A concept of suboptimal policy is introduced and numerical experiments are performed to test suboptimal policy suggested from numerical solution. The system behavior under the suboptimal policy is also discussed.

I. Introduction

A. *Mathematical Descriptions of General Control Systems* [2]

LET us denote by S a dynamic system of n degrees of freedom. Then, a set of n scalar functions of time t, $x^i(t)$, $i=1, 2, \cdots, n$ can describe completely the state of S at each time instant, *i.e.*, given $x^i(t)$, $i=1, 2, \cdots, n$ for $t \leq t_0$, the behavior of S with time for $t > t_0$ is completely described. These n functions of time can be regarded as n components of a vector function of time $x(t)$, called the *state vector* of the system S.

Although in a most general case the value of $x(t)$ for $t \geq t_0$ is dependent not only on the current state vector $x(t_0)$ but also on all the past values of the state vector $x(t)$, $t \leq t_0$ [1], in the following only such systems whose future state vectors $x(t)$ are completely and uniquely determined by their current state vector $x(t_0)=c$ are considered. That is to say, under suitable existence and uniqueness conditions, $x(t)$ becomes a function of time t and of the initial condition vector c, $x(t)=x(c, t)$, $x(0)=c$, taking $t_0=0$ without loss of generality.

From the uniqueness property which $x(t)$ satisfies, there follows

$$x(c, t_1 + t_2) = x(x(c, t_1), t_2), \tag{1}$$

which is nothing more than a mathematical representation of the *principle of causality* [2].

Let us take a unit time interval Δt and set

$$x(c, \Delta t) = T(c), \tag{2}$$

i.e., the initial state vector c is transformed into a new state vector $T(c)$, after the lapse of a unit time interval. Then from (1),

$$x(c, n\Delta t) = T(T(\cdots T(c) \cdots)) = T^n(c), \tag{3}$$

which is to say, the state vector at $t=n\Delta t$ is given by $T^n(c)$, the nth iterate of the function $T(c)$. This means that the time behavior of the deterministic dynamic system S at time $t=n\Delta t$ is determined by the successive

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iterates of a specific function of x, or successive point transformations on c [2]. Here, the kind of transformation that is applied on the state vector at $t=n\Delta t$, to get a new state vector at the next time instant $t=(n+1)\Delta t$, is independent of $x(n\Delta t)$. On the other hand, feedback control systems are designed in such a way as to utilize more than one type of transformation as a function of the state vectors [3], [4].

Let $t_1, t_2, \cdots$ be a sequence of times, $t_1<t_2<\cdots$ at which the system S is subject to a change in the form of a transformation, then

$$x(t_{k+1}) = T(x(t_k), v_k), \tag{4}$$

where the *control vector* v_k is included to indicate the fact that the transformation is dependent upon the choice of the control vector.

The choice is made in such a way as to optimize some performance index assigned to the control system [5], [6].

For example the state vector of a linear sampling control system in which the control vector enters linearly is given by

$$x_{k+1} = Ax_k + Bv_k + r_k, \tag{5}$$

where

Δt is the sampling time interval,
$x_k=x(k\Delta t)$ is an I-dimensional state vector,
A is I by I matrix,
B is I by J matrix,
v_k is J-dimensional control vector at $t=k\Delta t$,
r_k is I-dimensional vector at $t=k\Delta t$.

The term r_k may represent the effect of a random forcing variable with Markovian property [7] on the system. Then the relation of (5) denotes a stochastic transformation on the state vector x_k.

More generally, the state vector at time $(n+1)\Delta t$ is given by

$$x_{n+1} = T(x_n, v_n, r_n), \qquad n = 0, 1, 2, \cdots \tag{6}$$

the type of transformation on x_n being governed by variables v_n and r_n.[1]

B. *Adaptive Control Systems*

In many control systems, there exists random effects of one kind or another. They may be due to noisy components in the systems or due to a noisy environment which disturbs the outputs of the system. Very often, we do not have complete information about the distribution functions of random variables. Thus in many control processes, decision-making devices or controllers are called upon to make decisions under various uncertainties. As the process unfolds, additional information may become available to the controller which may shed light on these uncertainties. The control system then has the possibility of "learning" to improve its performance based upon experience, *i.e.*, the decision-making device may "adapt" itself to its environment.

In this paper, a class of distribution functions is considered in connection with certain time-discrete adaptive control systems, using a mathematical model of Bellman and Kalaba [2]. Distribution functions for random variables are assumed to be members of a certain parameter space, which is assumed to be discrete and finite. Therefore, a random variable R has the distribution function of the form $F(r, \alpha)$, where

$$P_r(\alpha = \alpha_i) = z_i \geq 0, \qquad i = 1, 2, \cdots, q$$

$$\sum_{i=1}^{q} z_i = 1.$$

Let us define the states of nature to be determined by these q possible values which the parameter α can assume. The state of nature is assumed not to change with time. Therefore, a control system must be able to estimate the state of nature in which it is likely to be operating.

When $z_i=1$ for some i, then the control system is known to be in the ith state of nature. That is to say, in this case, it is known that the random variable R has the distribution function $F(r, \alpha_i)$. This situation will be referred to as stochastic to distinguish it from the following situation. If $0<z_i<1$ for all $i=1, 2, \cdots, q$, then the control system is called adaptive.

The fact that this situation is really adaptive in the sense described at the beginning of the section can be seen as follows:

In the adaptive situation, there is a positive probability that the random variable R has the distribution function $F(r, \alpha_i)$, $i=1, 2, \cdots, q$.

Since an optimal choice of control vector v at each time instant depends on the knowledge of the distribution function of R, here the system is called upon to produce control vectors without having exact knowledge on the distribution function of R. It is also assumed that the *a priori* probability z_i is modified after each control action, as will be shown later, to produce *a posteriori* probability z_i', $i=1, 2, \cdots, q$. That is to say, the current estimate on the parameter α is updated at the conclusion of each control action.

II. Final-Value Systems: General Discussion

Let us take as the performance index of a control system some function ϕ of the state vector x_N at the time instant t_N, at which time the control action is assumed to terminate. The time t_N is called the final time. Control systems with such a performance index are called final-value systems [8]–[10]. The control system S must decide to apply v which minimizes the estimated expected value of $\phi(x_N)$, given the present state variable x, the number of the remaining decision stages n and the

[1] Eq. (6) means that once the system is in the state x_n, regardless of how the system reaches the state x_n, the next state x_{n+1} is given by (6). The way of looking at the next state and an optimal control vector as functions of current state vector, and not as functions of the initial vector, is essential in dealing with nondeterministic control systems [4], [6].

present estimate of the probability that the system is operating in the ith state of nature, z_i, $i=1, \cdots, q$.

The problem is adaptive, because the control device must guess or learn the state of nature and act accordingly, and this estimate as to the actual state of nature must be revised at the conclusion of each decision stage depending on the history of the observations of the random force r.

For the ease of presentation, let us take the Bernoulli random variable

$$r_n = \begin{cases} +c \text{ with probability } p_j \\ -c \text{ with probability } 1 - p_j,\ j = 1, 2 \end{cases} \tag{7}$$

$$\Pr(p = p_1) = z,\ \Pr(p = p_2) = q - z,\ p_1 > p_2.$$

That is, p is now the parameter α of the distribution function $F(r, \alpha)$ and nature is assumed to be in one of two possible states, H_1 and H_2. The random force r is assumed to be independently and identically distributed for each of the N stages.

Let us define the following:

$h_n(x; p_i)=$ the expected value of $\phi(x_N)$ employing the optimal policy, given the present state variable x of the system, and the number of the remaining decision stages n, and given that the state of nature is the ith state, $i=1, 2$. (8)

If we know definitely that nature is in the ith state (*i.e.*, $z=1$ or 0), then the optimal control force at each stage is given by the v_n which is obtained by solving the recurrence relation (9) derived by the application of the principle of optimality [2].

$$\begin{aligned} h_1(x; p_i) &= \min_{v_0} \left[p_i\phi(x^+) + (1 - p_i)\phi(x^-)\right] \\ h_n(x; p_i) &= \min_{v_{n-1}} \left[p_i h_{n-1}(x^+; p_i) + (1 - p_i)h_{n-1}(x^-; p_i)\right] \quad (9) \\ &\qquad n = 2, 3, \cdots, N, \\ &\qquad i = 1, 2, \end{aligned}$$

where x^+ and x^- are the state variable of S at the next decision stage, when the present state variable is x, with $r=+c$ and $r=-c$ respectively.

When we are given only the *a priori* probability as to nature being in the state H_1 or H_2, (9) must be rewritten by defining $k_n(x, z)$ to be the quantity corresponding to $h_n(x; p_i)$ given the present estimate z that the nature is in H_1. Noting that the *a posteriori* probability at the nth stage becomes the *a priori* probability for the next, $(n-1)$st, stage,

$$\begin{aligned} k_1(x, z) &= \min_{v_0} \{z[p_1\phi(x^+) + (1 - p_1)\phi(x^-)] \\ &\qquad + (1 - z)[p_2\phi(x^+) + (1 - p_2)\phi(x^-)]\}, \\ k_n(x, z) &= \min_{v_{n-1}} \{z[p_1k_{n-1}(x^+, z') + (1 - p_1)k_{n-1}(x^-; z'')] \\ &\qquad + (1 - z)[p_2k_{n-1}(x^+, z') \\ &\qquad\qquad + (1 - p_2)k_{n-1}(x^-, z'')]\}, \quad (10) \end{aligned}$$

where z' and z'' are given by (11) and (12) respectively.

If the present estimate of nature's being in H_1 is z, and if $r=c$ is realized, then the *a posteriori* probability that nature is in H_1 will become

$$z' = \frac{p_1 z}{zp_1 + (1 - z)p_2} = \frac{1}{1 + \dfrac{1 - z}{z}\alpha_1}. \tag{11}$$

When $r=-c$ is observed at this stage, the *a posteriori* probability becomes

$$z'' = \frac{z(1 - p_1)}{z(1 - p_1) + (1 - z)(1 - p_2)} = \frac{1}{1 + \dfrac{1 - z}{z}\alpha_0}, \tag{12}$$

where α_0 and α_1 are the likelihood ratios of H_1 over H_2, after $r_n=-c$ and $r_n=+c$ are observed, nameiy

$$\alpha_0 = \frac{1 - p_2}{1 - p_1}, \qquad \alpha_1 = \frac{p_2}{p_1}.$$

After n such observations, the *a priori* probability z becomes

$$z_n = \frac{1}{1 + \dfrac{1 - z}{z}\alpha_n}, \tag{13}$$

where α_n is the likelihood ratio of the n observations, *i.e.*, if $r=c$ are observed n_1 times, and $r=-c$ are observed n_2 times, then

$$\alpha_n = \left(\frac{p_2}{p_1}\right)^{n_1}\left(\frac{1 - p_2}{1 - p_1}\right)^{n_2}, \qquad n_1 + n_2 = n. \tag{14}$$

From (10),

$$\begin{aligned} k_1(x, z) &= \min_{v_0} \{[zp_1 + (1 - z)p_2]\phi(x^+) \\ &\qquad + [z(1 - p_1) + (1 - z)(1 - p_2)]\phi(x^-)\} \\ k_n(x, z) &= \min_{v_{n-1}} \{[zp_1 + (1 - z)p_2]k_{n-1}(x^+, z') \\ &\qquad + [z(1 - p_1) + (1 - z)(1 - p_2)]k_{n-1}(x^-, z'')\} \\ &\qquad n = 2, 3, \cdots, N. \quad (15) \end{aligned}$$

Let us note that if $z=1$ or $z=0$, then $z'=z''=1$ or $z'=z''=0$, and (10) and (15) reduce to (9), since $k_n(x, 1)=h_n(x; p_1)$, $k_n(x, 0)=h_n(x; p_2)$.

If one could obtain, before solving (15), some information on the functional structure of $k_n(x, z)$ from the knowledge of $h_n(x; p_1)$ and $h_n(x; p_2)$, then one would be in a better position to devise an appropriate computational procedure and/or an analytic approximation of solving (15).

One such structural information is given by the following.

Proposition

$k_n(x, z)$ of (10) and (15) is concave in z, *i.e.*,

$$k_n(x, z) \geq zk_n(x, 1) + (1 - z)k_n(x, 0), \quad n = 1, 2, \cdots \quad (16)$$

For the proof, see Appendix I.

Eq. (16) supplies the lower limit on $k_n(x, z)$, given $k_n(x, 1) = h_n(x, p_1)$ and $k_n(x, 0) = h_n(x, p_2)$.

The upper limit on $k_n(x; z)$ is given from (10) by

$$\begin{aligned} k_n(x, z) &\leq \min_{v_{n-1}} \max [p_1 k_{n-1}(x^+, z') + (1 - p_1)k_{n-1}(x^-, z''), \\ &\qquad p_2 k_{n-1}(x^+, z') + (1 - p_2)k_{n-1}(x^-, z'')] \\ &\leq \min_{v_{n-1}} \max [k_{n-1}(x^+, z'), k_{n-1}(x^-, z'')] \\ &\triangleq ct_n(x, z). \end{aligned}$$

Observe that $ct_n(x, z)$ requires less computation than $k_n(x, z)$.

Thus, we know that the minimum of the estimated expected value of $\phi(x_N)$ for this type of adaptive final value system, $k_n(x, z)$, is concave in z, and (16) provides a lower bound on $k_n(x, z)$, given the expected values of the criterion function for the corresponding stochastic case, $h_n(x; p_1)$ and $h_n(x; p_2)$.

Let us note that in order to prove the concavity (16), the actual form of the difference equation of the system (6), distribution function $F(r, \alpha)$ and ϕ are immaterial. The concavity is a general characteristic of final value control systems with a finite number of states of nature. Since the knowledge of a sequence of optimal v is equivalent to that of the criterion functional values [2], the lower bound of $k_n(x, z)$ may be used to derive initial approximate policy to start a sequence of approximations in policy space [11].

Let us assume for the moment that the recurrence relation (10) has been solved, and let us consider certain fixed values of x and n. Then in (10) let us define s_1 and s_2 as

$$s_i = p_i k_{n-1}(x^+, z') + (1 - p_i)k_{n-1}(x^-, z''), \quad i = 1, 2, \quad (18)$$

thus s_1 and s_2 will be in general functions of x, n, v and z.

We can now write (10) as

$$\begin{aligned} k_n(x, z) &= \min_{v} [zs_1 + (1 - z)s_2] \\ &= zs_1^* + (1 - z)s_2^*, \end{aligned} \quad (19)$$

where s_1^* and s_2^* are s_1 and s_2 for a certain v which is optimal for x and n considered.

If one regards s_1 and s_2 as the loss of the control system with v when nature is H_1 and H_2 respectively,[2] then the control process may be considered to be the S game [12] where nature has two strategies. When nature employs a mixed strategy with probability distribution $\mathfrak{z} = (z, 1-z)$, the expected value of the loss to the control system with v becomes

$$zs_1 + (1 - z)s_2, \quad (20)$$

and the $s^* = (s_1^*, s_2^*)$ is the minimum of (20).

Viewed in the light of game theory, the use of the *a priori* and the *a posteriori* probability distributions in (10) seems to be quite natural. This can be formalized by assuming that z will be transformed by the Bayes formula.

The reason for introducing the concept of S game here is that there exist mathematical theories on set-theoretical relations among the classes of strategies, and they are useful in discussing optimal strategies or optimal policies. Although it does not seem possible to fit questions in adaptive control processes completely in the existing frame of the theory of statistical decisions and sequential analysis, certain analogies can be used to advantage in the construction of the theory of adaptive control processes, and in deriving approximate solutions for functional equations derived by the application of the principle of optimality.

When z, z', and z'' are close to each other, it is reasonable to expect that $k_n(x, z)$, $k_n(x, z')$ and $k_n(x, z'')$ are also close to one another, since they are continuous in x and z. The recurrence relation defining $\bar{k}_n(x, z)$ by

$$\begin{aligned} \bar{k}_1(x, z) &= k_1(x, z) \\ \bar{k}_n(x, z) &= \min_{v_{n-1}} \{z[p_1 \bar{k}_{n-1}(x^+, z) + (1 - p_1)\bar{k}_{n-1}(x^-, z)] \\ &\qquad + (1 - z)[p_2 \bar{k}_{n-1}(x^+, z) + (1 - p_2)\bar{k}_{n-1}(x^-, z)]\} \\ &\qquad n = 2, 3, \cdots, N, \end{aligned} \quad (21)$$

may be used as an approximation to the exact recurrence relation (10).

III. Structure Dependence of Quadratic Criterion Function of Final Value Control Systems on the Parameter z.

Let us take a quadratic form for $\phi(x_N)$ as

$$\phi(x_N) = (x_N', PxN), \quad (22)$$

where P is a positive semi-definite matrix, x_N is the final position of the system, and x_N' is the transpose of x_N.

Let us assume that the system is to operate under the constraint

$$\sum_{n=0}^{N-1} (v_n', Dv_n) \leq K, \quad (23)$$

where D is a positive definite matrix and v_n is the control variable of the system at the nth decision stage and v_n' its transpose, and it is assumed that $-\infty < v_n < +\infty$, $n = 0, 1, \cdots, N-1$.

The problem can then be formulated to minimize the expected value of

$$J_N = (x_N', Px_N) + \lambda \sum_{n=0}^{N-1} (v_n', Dv_n), \ \lambda > 0, \quad (24)$$

where λ is the Lagrange multiplier [6], [13], [18].

[2] This is exactly true only for $z=1$ or $z=0$.

The criterion function of the system is given by

$$f_N(x_0, z) = \min_{v_0, v_1, \cdots, v_{N-}} E(J_N), \tag{25}$$

where x_0 is the initial position of the system, z is the *a priori* distribution of the states of nature between H_1 and H_2 and $v_0, v_1, \cdots, v_{N-1}$ are to be chosen optimally, *i.e.*, to realize the minimum value of $E(J_N)$ taken with respect to the random force r and z.

Let us assume further that if the state variable of the system at the present decision stage is x, then the state variable at the next decision stage will be given either by

$$x_+ = Ax + B_+v, \quad \text{or by} \quad x_- = Ax + B_-v, \tag{26}$$

where suffixes $+$ and $-$ refer to the two possible states of the random variable r given by (7) to which the system is subjected. Here, the effect of r is, therefore, assumed to appear in two possible B's, *i.e.*, B_+ and B_-.

For ease of presentation, let us assume here that the state variable of the system is one-dimensional, and carry out the discussion in terms of scalar quantities. Eq. (24) is rewritten as

$$J_N' = J_N/p = x_N^2 + \lambda \sum_{n=0}^{N-1} v_n^2, \tag{24'}$$

where $\lambda d/p$ is renamed as λ, $\lambda > 0$.

The criterion function is now taken to be

$$f_N(x_0, z) = \min_{v_0, v_1, \cdots, v_{N-1}} E(J_N'), \tag{25'}$$

since the minimization of (24) is equivalent to that of (24').

The new state variable of the system is related to the old state variables by

$$x_+ = ax + b_+v, \quad \text{and} \quad x_- = ax + b_-v, \tag{26'}$$

where $0 < a < 1$.

The functional equation is given similarly to (15) by

$$f_k(x, z) = \min_{v_{k-1}} \{[zp_1 + (1-z)p_2] f_{k-1}(x_+, z') + [(1-p_1)z + (1-p_2)(1-z)] f_{k-1}(x_-, z'') + \lambda v_{k-1}^2\}$$
$$k = 1, \cdots, N, \tag{27}$$

$$f^1(x, z) = \min_{v_0} \{[zp_1 + (1-z)p_2](ax + b_+v_0)^2 + \lambda v_0^2 + [(1-p_1)z + (1-p_2)(1-z)](ax + b_-v_0)^2\}, \tag{28}$$

where

$$z' = \frac{p_1 z}{p_1 z + (1-z)p_2}, \quad z'' = \frac{(1-p_1)z}{(1-p_1)z + (1-p_2)(1-z)}.$$

From (28), $(\partial f_1/\partial v_0) = 0$ gives the minimizing v_0. The existence of the minimum is assured by the fact that $f_1(x, z)$ is convex quadratic in v_0.

This v_0 is given by

$$v_0 = g_0(z) \cdot x, \tag{29}$$

where

$$g_0(z) \triangleq -aE(b)/(\lambda + E(b^2)),$$
$$E(b) \triangleq zE_1(b) + (1-z)E_2(b), \text{ and}$$
$$E_i(b) \triangleq p_i b_+ + (1-p_i)b_-, \quad i = 1, 2.$$

and $E(b^2)$ is defined similarly to $E(b)$.

The criterion function is given by

$$f_1(x, z) = w_1(z) \cdot x^2, \tag{30}$$

where

$$w_1(z) = a^2[1 - E(b)^2/(\lambda + E(b^2))]. \tag{31}$$

It can be shown inductively that, in general, v_n is linear in x and $f_n(x, z)$ is quadratic in x and can be expressed by

$$f_n(x, z) = w_n(z)x^2,$$
$$v_n(x, z) = g_n(z)x. \tag{32}$$

The recurrence relations for $w_n(z)$ and $g_n(z)$ are given by

$$w_{n+1}(z) = a^2\left[E(w_n(z)) - \frac{E(bw_n(z))^2}{\lambda + E(b^2w_n(z))}\right]$$
$$n = 1, 2, \cdots, N-1. \tag{33}$$

$$g_n(z) = -a\frac{E(bw_n(z))}{\lambda + E(b^2w_n(z))} \quad n = 1, 2, \cdots, N. \tag{34}$$

The symbols appearing in (33) and (34) are defined as follows:

$$E(w_n(z)) = zE_1(\omega_n(z)) + (1-z)E_2(w_n(z)), \tag{35}$$

where

$$E_i(w_n(z)) = p_i w_n(z') + (1-p_i)w_n(z''), \qquad i = 1, 2.$$

Similarly,

$$E(bw_n(z)) = zE_1(bw_n(z)) + (1-z)E_2(bw_n(z)), \tag{36}$$

where

$$E_i(bw_n(z)) = p_i b_+ w_n(z') + (1-p_i)b_- w_n(z''), \quad i = 1, 2;$$

and

$$E(b^2w_n(z)) = zE_1(b^2w_n(z)) + (1-z)E_2(b^2w_n(z)), \tag{37}$$

where

$$E_i(b^2w_n(z)) = p_i b_+^2 w_n(z') + (1-p_i)b_-^2 w_n(z''), \; i = 1, 2.$$

Now that the recurrence relation on w_n is given, let us investigate the functional structures of $g_n(z)$ and $w_n(z)$, namely their dependence on z. Assume

$$0 \leq p_2 \leq p_1 \leq 1, \; b_- \leq b_+ \quad \text{and} \quad b_-^2 \leq b_+^2, \tag{38}$$

the case where $b_- > b_{++}$ can be easily taken care of in the similar manner.

Eq. (29), when written out in detail, becomes

$$g_0(z) = -a\{E_2(b) + [E_1(b) - E_2(b)]z\}/\{\lambda + E_2(b^2) + [E_1(b^2) - E_2(b^2)]z\}, \quad (29')$$

where

$$E_1(b) - E_2(b) = (p_1 - p_2)\cdot(b_+ - b_-) \geq 0. \quad (39)$$

Similarly

$$E_1(b^2) - E_2(b^2) = (p_1 - p_2)(b_+{}^2 - b_-{}^2) \geq 0. \quad (40)$$

Since

$$\left(\frac{dg_0(z)}{dz}\right) = -a\frac{[E_1(b)-E_2(b)][\lambda+E_2(b^2)]-E_2(b)[E_1(b^2)-E_2(b^2)]}{\{\lambda+E_2(b^2)+[E_1(b^2)-E_2(b^2)]z\}^2}$$

does not change sign for $0 \leq z \leq 1$, $g_0(z)$ is monotonic in z. For λ sufficiently large,[3] $g_0(z)$ is monotonically decreasing.

From (31),

$$w_1(z) = a^2\{1 - [E_2(b) + (E_1(b) - E_2(b))z]^2/[\lambda + E_2(b^2) + (E_1(b^2) - E_2(b^2))z]\}. \quad (41)$$

If one specifies $b_\pm$ to be

$$b_\pm = \pm s, \quad (42)$$

then, since $E_i(b^2) = s^2$, $i = 1, 2$, $w_1(z)$ becomes a concave quadratic function in z.

Or, even without assuming (42), if λ is large, then $w_1(z)$ will be concave quadratic in z, neglecting terms $0(1/\lambda^2)$.[4]

In the following, let us consider the situation where $E(b^2)/\lambda \ll 1$ holds.[5] Loosely speaking, this means that only a small amount of energy is available for control purposes, since λ can be interpreted as price of control as mentioned in footnote 3.

From (41),

$$w_1(z) = a^2\left\{1 - \frac{1}{\lambda}[E_2(b) + (E_1(b) - E_2(b))z]^2\right\} + 0(1/\lambda^2). \quad (43)$$

As a first approximation, take

$$w_1(z) = a^2 < 1. \quad (44)$$

Then, from (33),

$$w_{n+1}(z) = a^{2(n+1)}. \quad (45)$$

This, in turn, gives from (34)

$$g_n(z) = -\frac{a^{2n+1}}{\lambda}\{E_2(b) + [E_1(b) - E_2(b)]z\}. \quad (46)$$

Therefore, $g_n(z)$, and consequently the optimal v_n are monotonically decreasing in z. Note that $g_n(z) = 0$ for the zero-th degree term in the expansion in $1/\lambda$.

Let us make (45) more precise by retaining terms of the order up to $1/\lambda$.

Then,

$$w_1(z) = a^2\left\{1 - \frac{1}{\lambda}[E_2(b) + [E_1(b) - E_2(b)]z]^2\right\}, \quad (43')$$

and

$$w_{n+1}(z) = a^2\left[E(w_n(z)) - \frac{1}{\lambda}E(bw_n(z))^2\right],$$
$$n = 2, \cdots, N-1. \quad (45')$$

As can be seen from (35) and (37), the right-hand side of (45′) contains $w_n(z')$ and $w_n(z'')$. They are expanded in Taylor series at z,[6]

$$w_n(z') = w_n(z) + \left(\frac{\partial w_n}{\partial z}\right)_z (z' - z) + \frac{1}{2}\left(\frac{\partial^2 w_n}{\partial z^2}\right)_z (z' - z)^2 + 0((z' - z)^3),$$

$$w_n(z'') = w_n(z) + \left(\frac{\partial w_n}{\partial z}\right)_z (z'' - z) + \frac{1}{2}\left(\frac{\partial^2 w_n}{\partial z^2}\right)_z (z'' - z)^2 + 0((z'' - z)^3). \quad (47)$$

Then, from (35)

$$E(w_n(z)) = w_k(z) + \left(\frac{\partial^2 w_k}{\partial z^2}\right)_z Z, \quad (48)$$

[3] It is true that with equality in (23), the value for λ will be uniquely determined.

In the dynamic programming approach to the use of Lagrange multipliers, however, it is more convenient to work with (24) or (24′) directly, treating λ as if it is another independent variable, rather than working with K in (23). (This is particularly true in computational solutions.)

For example, f_n and v_n are tentatively determined by assigning a certain value to λ. Inserting these v_n back in (23), if it is found that (23) is not satisfied, then the value of λ is increased and the above process will be repeated. Thus λ can usually be interpreted as price or cost.

Parameter studies on K can be replaced by equivalent parameter studies on λ. The assumption on the range of K can be replaced by an equivalent assumption on the range of λ. This is what is done in the paper.

The theoretical justification for the above statement is required and can be found in Ch. 4 of [18].

[4] The symbol $0(x)$ means a quantity of the order x, thus $0(1/\lambda^2)$ indicates a quantity which approaches zero essentially at the rate of $1/\lambda^2$, *i.e.*,

$$\frac{0\left(\frac{1}{\lambda^2}\right)}{\frac{1}{\lambda^2}} \to \text{constant as } \lambda \to \infty.$$

[5] Asymptotic expansions in terms of $1/\lambda$ are therefore investigated.

[6] At $z = 1$, the derivative from the left is used. At $z = 0$, the derivative from the right is used.

where

$$2Z = (p_1 - p_2)^2 z^2 (1-z)^2 / \{[zp_1 + (1-z)p_2] \cdot [(1-p_1)z + (1-p_2)(1-z)]\}.$$

$$Z \simeq (p_1 - p_2)^2 z^2 / 2p_2(1-p_2), \text{ if } z \text{ is close to zero.} \quad (49)$$

If z is close to one, then

$$Z \simeq (p_1 - p_2)^2 (1-z)^2 / 2p_1(1-p_1). \quad (49')$$

Following the similar derivation,

$$E(bw_n(z)) \simeq w_n(z)E(b) + \left(\frac{\partial w_n}{\partial z}\right) z(1-z)[E_1(b) - E_2(b)] + \left(\frac{\partial_2 w_n}{\partial z^2}\right) Z', \quad (50)$$

where

$$2Z' = [p_1 z + (1-z)p_2] b_+ (z'' - z)^2 + [(1-p_1)z + (1-p_2)(1-z)] b_- (z'' - z)^2$$

When z, z' and z'' are close to each other, Z and Z' can be approximately made zero.

From (43′), (45′), (48) and (50),

$$w_2(z) \simeq a^4 \left\{1 - \frac{1}{\lambda} [E_2(b) + (E_1(b) - E_2(b))z]^2 + 2(E_1(b) - E_2(b)) \cdot Z + a^2 E(b)^2 \right\}.$$

Since in the neighborhood of $z=0$ and $z=1$, Z is quadratic in z, $w_2(z)$ is quadratic in z in the same neighborhood. Also $w_n(z)$ will be quadratic in z, in the same neighborhood as can be seen inductively, if the terms up to $1/\lambda$ are retained.

Differentiating (45′) twice and making use of (48) and (49)

$$\frac{\partial^2 w_{n+1}(z)}{\partial z^2} \simeq a^2 \left\{ \left(1 + \frac{\partial^2 Z}{\partial z^2}\right) \left(\frac{\partial^2 w_n}{\partial z^2}\right) - \frac{1}{\lambda} 2a^{4n} (E_1(b) - E_2(b))^2 \right\}; \quad (51)$$

i.e.

$$\left.\frac{\partial^2 w_{n+1}}{\partial z^2}\right|_{0,1} = z^2 (1 + A_{0,1}) \left(\frac{\partial^2 w_n}{\partial z^2}\right)_{0,1} - \frac{2a^{4n+2}}{\lambda} (E_1(b) - E_2(b))^2,$$

where

$$A_0 = \left.\frac{\partial^2 Z}{\partial z^2}\right|_0 = \frac{(p_1 - p_2)^2}{p_2(1-p_2)}, \quad A_1 = \left.\frac{\partial^2 Z}{\partial z^2}\right|_1 = \frac{(p_1 - p_2)^2}{p_1(1-p_1)}.$$

Since

$$\frac{\partial^2 w_1}{\partial z^2} = -\frac{2a^2}{\lambda} (E_1(b) - E_2(b))^2,$$

$$\left.\frac{\partial^2 w_2}{\partial z^2}\right|_{0,1} = -\frac{2a^4}{\lambda} (E_1(b) - E_2(b))^2 [(1 + A_{0,1}) + a^2].$$

In general,

$$\left.\frac{\partial^2 w_{k+1}}{\partial z^2}\right|_{0,1} = -\frac{2a^{2k}}{\lambda} (E_1(b) - E_2(b))^2 \cdot [\rho_{0,1}{}^k + a^2 \rho_{0,1}{}^{k-1} + \cdots + a^{2k}]$$
$$= -\frac{2(a^2 \rho_{0,1})^k}{\lambda} (E_1(b) - E_2(b))^2 \cdot \frac{1 - \left(\dfrac{a^2}{\rho_{0,1}}\right)^{k+1}}{1 - \left(\dfrac{a^2}{\rho_{0,1}}\right)}, \quad (52)$$

where

$$\rho_{0,1} = 1 + A_{0,1}.$$

Therefore, in the neighborhood of $z=0, 1$,

$$w_{n+1}(z) = \frac{(a^2 \rho_{0,1})^n}{\lambda} (E_1(b) - E_2(b))^2 \cdot \frac{1 - \left(\dfrac{a^2}{\rho_{0,1}}\right)^{n+1}}{1 - \left(\dfrac{a^2}{\rho_{0,1}}\right)} z(1-z) + w_{n+1}(0)(1-z) + w_{n+1}(1)z. \quad (53)$$

If $z=1$, then $z'=z''=1$. Therefore, from (35)–(37),

$$E(w_n(1)) = E_1(w_n(1)) = p_1 w_n(1) + (1-p_1) w_n(1) = w_n(1).$$

$$E(bw_n(1)) = E_1(bw_n(1)) = (p_1 b_+ + (1-p_1) b_-) w_1(1) = E_1(b) w_n(1).$$

$$E(b^2 w_n(1)) = E_1(b^2 w_n(1)) = (p_1 b_+{}^2 + (1-p_1) b_-{}^2) w_n(1) = E_1(b^2) w_n(1).$$

From (33),

$$w_{n+1}(1) \simeq a^2 \left[1 - \frac{1}{\lambda} \cdot E_1(b)^2 w_n(1)\right] \cdot w_n(1)$$
$$n = 1, \cdots, N,$$
$$w_1(1) = a^2 [1 - E_1(b)^2 / (\lambda + E_1(b^2))]. \quad (54)$$

Similarly, if $z=0$, then $z'=z''=0$ and

$$w_{n+1}(0) \simeq a^2 \left[1 - \frac{E_2(b)^2 w_n(0)}{\lambda}\right] \cdot w_n(0),$$
$$n = 1, \cdots, N,$$
$$w_1(0) = a^2 [1 - E_2(b)^2 / (\lambda + E_2(b^2))]. \quad (55)$$

From (32) and (53),

$$g_n(z) \simeq -a/\lambda \cdot [E(b)w_n(1)z + E(b)w_n(0)(1-z) + z(1-z)(E_1(b) - E_2(b)) \cdot (w_n(1) - w_n(0))] + 0(1/\lambda^2). \quad (56)$$

Since its derivative with respect to z does not change sign as seen from

$$\frac{dg_n}{dz} = -(a/\lambda)[(E_1(b) - E_2(b))w_n(0) + E_1(b)(w_n(1) - w_n(0))], \quad (57)$$

$g_n(z)$ is monotonic in z.

More generally, if terms of only up to $1/\lambda$ are retained, since

$$E(bw_n(z)) = E(b)a^{2n} + 0(1/\lambda), \quad (58)$$

$$g_n(z) = -a^{2n+1}/\lambda[E_2(b) + (E_1(b) - E_2(b)E_1(b)E_2(b))z] + 0(1/\lambda^2), \quad (59)$$

$g_n(z)$ is monotonically decreasing in z.

From (32), the optimal $v_n(x, z)$ is, therefore, monotonically decreasing in z under the conditions stated above.

Thus, $f_n(x, z)$ is quadratic in the neighborhood of $z=1$ and 0 and concave over $0<z<1$.

IV. Contactor Servo Final Value Systems

Let us next consider the situation where a constraint on v is imposed as

$$v = \begin{cases} +m \\ \text{or} \qquad m > c, \\ -m \end{cases}$$

and let us consider the difference equation

$$x_{n+1} = \alpha x_n + r_n + v_n, \quad (60)$$

$$r_n = \begin{cases} +c & \text{with probability } p \\ -c & \text{with probability } 1-p, \end{cases} \quad (7')$$

where p is assumed to be known.

The criterion function is taken to be

$$\phi(x_N) = x_N^2. \quad (61)$$

The state variable x_n is taken to be a scalar.

The functional equation describing the control process is given similarly to (9) by replacing p_i by p,

$$\begin{cases} h_1(x) = \min\limits_{v=\pm m} [px_+^2 + (1-p)x_-^2] \\ h_k(x) = \min\limits_{v=\pm m} [ph_{k-1}(x_+) + (1-p)h_{k-1}(x_-)] \end{cases} \quad (9')$$

$$k = 2, \cdots$$

where

$$x_+ = \alpha x + c + v, \quad x_- = \alpha x - c + v, \quad (62)$$

from (60).

Since the equation governing the system is fairly simple, a certain amount of analytic manipulation of the criterion function is possible.

From (9′) and (62) one gets

$$h_1(x) = \min_{v_1=\pm m} [4p(1-p)c^2 + (\alpha x + (2p-1)c + v_1)^2]. \quad (63)$$

Therefore, the optimal control force is given by

$$\begin{aligned} v_1 &= -m; \quad \text{if } \alpha x + (2p-1)c \geqq 0, \\ v_1 &= m; \quad \text{if } \alpha x + (2p-1)c \leqq 0, \end{aligned} \quad (64)$$

and

$$h_1(x) = 4p(1-p)c^2 + [\alpha x + (2p-1)c + v_1]^2. \quad (63')$$

Similarly,

$$h_2(x) = [ph_1(x_+) + (1-p)h_1(x_-)]v_2 \quad (65)$$

where

$$x_+ = \alpha x + c + v_2, \qquad x_- = \alpha x - c + v_2,$$

and v_2 is understood to take m or $-m$, whichever is the optimal choice in that it minimizes $h_2(x)$.

From (63′),

$$\begin{aligned} h_1(x_+) &= 4p(1-p)c^2 + [\alpha^2 x + \alpha c + \alpha v_2 + (2p-1)c + v_1^+]^2, \\ h_1(x_-) &= 4p(1-p)c^2 + [\alpha^2 x - \alpha c + \alpha v_2 + (2p-1)c + v_1^-]^2, \end{aligned} \quad (66)$$

where v_1^+ and v_1^- represent respectively the optimal choice of v_1, when the initial position is x_+ and x_- respectively.

From (65) and (66),

$$\begin{aligned} h_2(x) = {} & 4p(1-p)c^2(1+\alpha^2) \\ & + [\alpha^2 x + \alpha v_2 + (2p-1)c(1+\alpha)]^2 \\ & + [v_1^2 + 2pv_1^+ - (1-p)v_1^-]\alpha c \\ & + 2(\alpha^2 x + \alpha v_2 + (2p-1)c)(pv_1^+ + (1-p)v_1^-). \end{aligned} \quad (67)$$

If $v_1^+ = v_1^-$, this becomes

$$h_2(x) = 4p(1-p)c^2(1+\alpha^2) + [\alpha^2 x + (2p-1)c(1+\alpha) + \alpha v_2 + v_1]^2. \quad (68)$$

This will be the case if $|x|$ is sufficiently large so that the outcome of r_2 does not affect the optimal choice of v_1. In other words, $|x|$ is sufficiently away from the switching boundary for v_1, so that two possible values of r_2 can not make the system straddle the switching boundary.

If $v_1^- = -v_1^+$, then (67) becomes

$$\begin{aligned} h_2(x) = {} & 4p(1-p)c^2(1+\alpha^2) \\ & + 8p(1-p)\alpha c v_1^+ + 4p(1-p)(v_1^+)2 \\ & + [\alpha^2 x + (2p-1)c(1+\alpha) + \alpha v_2 + (2p-1)v_1^+]^2. \end{aligned} \quad (67')$$

The value of v_2 is determined by

$$v_2 = -m \operatorname{sgn}(\alpha^2 x + (2p-1)c(1+\alpha) + v_1); \qquad \text{if } v_1^- = v_1^+,$$

$$v_2 = -m \operatorname{sgn}(\alpha^2 x + (2p-1)c(1+\alpha) + (2p-1)v_1^+); \qquad \text{if } v_1^- = -v_1^+. \tag{69}$$

The general structure of $h_N(x)$ is seen to be given by

$$\begin{aligned} h_N(x) = {} & 4p(1-p)[c^2 + (\alpha c + \epsilon_1 v_1^+)^2 + \cdots \\ & + (\alpha^{N-1}c + \epsilon_{N-1}\alpha^{N-2}v_{N-1}^+)^2] \\ & + [\alpha^N x + (2p-1)(1+\alpha+\cdots+\alpha^{N-1}) + \gamma_1 \\ & + \alpha\gamma_2 + \cdots + \alpha^{N-2}\gamma_{N-1} + \alpha^{N-1}v_N]^2, \end{aligned} \tag{70}$$

where

$$\epsilon_i = \begin{cases} 0; & \text{if } v_i^- = v_i^+, \\ 1; & \text{if } v_i^- = -v_i^+, \end{cases}$$

$$\gamma_i = \begin{cases} v_i; & \text{if } v_i^- = v_i^+ \\ (2p-1)v_i^+; & \text{if } v_i^- = -v_i^+, \end{cases}$$

and v_i^+ and v_i^- are defined similarly to v_1^+ and v_1^-. The truth of this formula can be seen by the application of mathematical induction.

The control force is given by

$$v_N = -m \operatorname{sgn}(\alpha^N x + (2p-1)c(1+\alpha+\cdots+\alpha^{N-1}) + \gamma_1 + \cdots + \alpha^{N-2}\gamma_{N-1}). \tag{71}$$

It should be noted here that although (70) gives the functional form of $h_N(x)$, the values of ϵ_i, γ_i, $i=1, 2, \cdots, N-1$ and v_N are not given explicitly, and that (70) does not serve to compute $h_N(x)$ as the function of x and N.

Let us now investigate a certain dependence of the policy and the criterion function on x and p. The dependence of the v_k and $h_k(x)$ on p is expressed as $v_k(x; p)$ and $h_k(x; p)$, *i.e.*, $v_k(x; p)$ will be either m or $-m$ depending on which gives smaller $h_k(x; p)$ at the point (x, p).

Arguing inductively, one sees

$$h_k(x; p) = h_k(-x; 1-p), \quad k = 1, 2, \cdots, N \text{ holds.} \tag{72}$$

The proof is given in the Appendix II.

The relation (72) is significant because it means the reduction by half of the amount of computation necessary for solving the functional equation (9′). This relation holds, even when v is extended from $v = \pm m$ to more than two possible values, so long as the values v can assume are symmetric with respect to $v=0$, as in $v=m$, 0, or $-m$.

The cases of $v = \pm m$ and $v = \pm m$, 0 are computed and $h_k(x)$ are found to satisfy $h_k(x; p) = h_k(-x; 1-p)$, $k=1, 2, \cdots, N$.

Looking at (70), one notices that the explicit dependence of the criterion function $h_n(x)$ on x decreases as n becomes large, and that once $h_k(x)$ becomes constant, then $h_n(x)$ will remain the same for all $n \geq k$.

The recurrence relation (9′) was solved computationally[7] with the boundary condition that $h_n(x; pi) = h_n(D; pi)$ for $x \geq D$ and $h_n(x; pi) = h_n(-D; p_i)$ for $x \leq -D$, $i=1, 2$. The essential steps in computational procedures are sketched in Appendix III [6].

Having computed $h_n(x; p_1)$ and $h_n(x; p_2)$, one can use them to provide a lower bound on the criterion function of the adaptive final value system $k_n(x, z)$ of (10) as discussed in Section II, where $z = \Pr \pm (p = p_1)$, $1 - z = \Pr(p = p_2)$.

Fig. 1 shows $k_n(x, z)$ as a function of z for various n and x. It is seen that for $n \geq 8$, the lower bound as a linear combination of $h_n(x; p_1)$ and $h_n(x; p_2)$ is a fairly good approximation to $k_n(x, z)$.

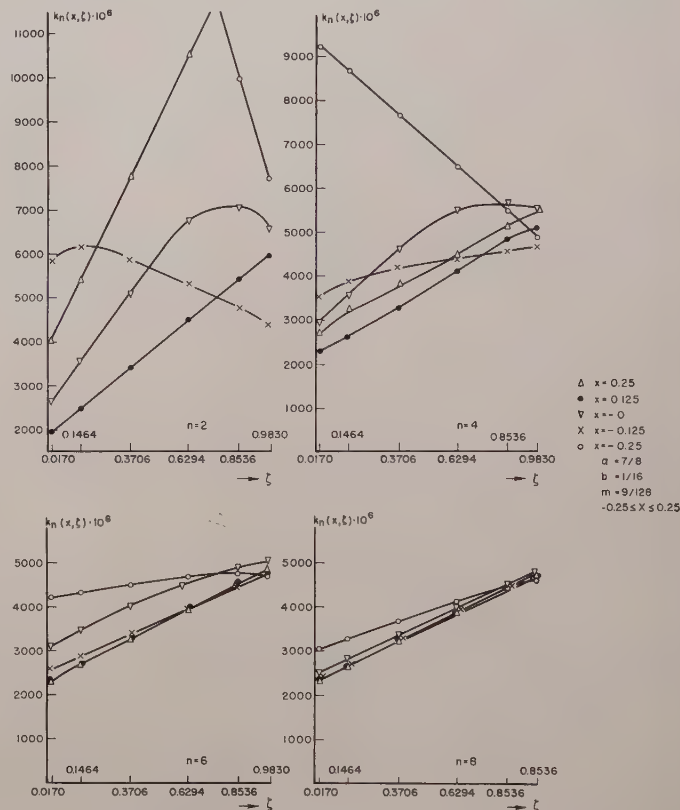

Fig. 1—The criterion function, $k_n(x, z)$, as the function of the *a priori* probability z.

V. One-Stage Suboptimal Policy

Fig. 2 shows an optimal control variable as a function of x and n for $p=0.625$, $D=\frac{1}{4}$, $\alpha=\frac{7}{8}$, $c=\frac{1}{16}$, $m=9/128$, $n \leq N = 12$.[8] Notice that the boundaries between $v_n(x; p) = +m$ and $v_n(x; p) = -m$ are not simple straight lines.

Given the total duration N of the control process, the optimal sequence of control vectors $\{v_n(x; p)\}$ are given by solving (9′) sequentially for $n = 1, 2, \cdots, N$.

[7] The computation was done on the IBM 709 at the Western Data Processing Center, University of California at Los Angeles.

[8] N is taken to be 12. For $N > 12$, the system stays pinned to either $x = D$ or $x = -D$, because of the particular boundary condition of the system until the number of remaining stages n becomes less than 12 or 11.

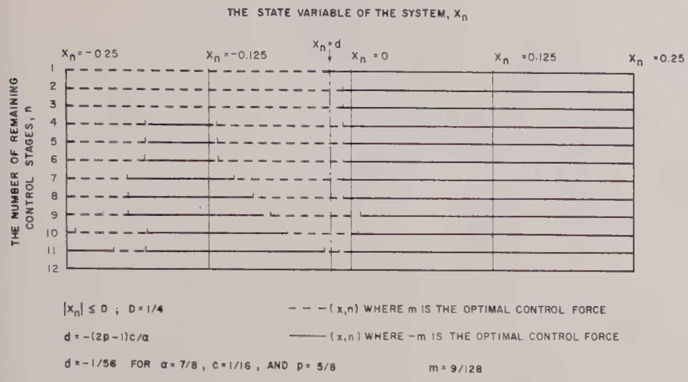

Fig. 2—Boundaries between $+m$ control force and $-m$ control force for the optimal policy for $p=0.625$.

When $N=1$, the optimal control vector is given by

$$v_1(x;\, p) = -m\cdot\operatorname{sgn}(\alpha x + (2p-1)c). \tag{64}$$

If (64) is used as $v_n(x;\, p)$ for $n \geq 2$, then it constitutes a suboptimal policy.

The switching boundary, then, is independent of n and is represented by a straight line given by

$$x = -(2p-1)c/\alpha \tag{73}$$

and shown in Fig. 2 as the line $x_n = d$. That is, according to this one stage suboptimal policy, $v_n = -m$ if $x_n \geq d$ and $v_n = +m$ if $x_n < d$.

From Fig. 2 it is noted that for $x>0$ the optimal and suboptimal policies agree fairly well, although the agreement is not particularly good for $x<0$.

It might be expected, therefore, that if the system starts from the initial position $x=D$, then the suboptimal policy is a fairly good approximation for the optimal policy.

To see to what extent the conjecture is correct, the Monte Carlo method [16] was used to simulate the system behavior from initial positions $x_0 = \pm D$. Forty simulated runs were made using random numbers to generate $r_n = +c$ and $r_n = -c$ with appropriate p. With $x_0 = D$

$$\left.\begin{aligned} E(x_N^2)_{\text{optimal}} &= 0.00367, \\ E(x_N^2)_{\text{suboptimal}} &= 0.00374, \end{aligned}\right\} \text{ for } p = 0.625.$$

With $x_0 = -D$,

$$\left.\begin{aligned} E(x_N^2)_{\text{optimal}} &= 0.00437, \\ E(x_N^2)_{\text{suboptimal}} &= 0.00441, \end{aligned}\right\} \text{ for } p = 0.625.$$

In spite of the relatively similar values in $E(x_N^2)$, the suboptimal policy for $x_0 = D$ appears to be better than that for $x_0 = -D$, as conjectured, by the fact that out of 40 trials with $x_0 = D$, the optimal and suboptimal policies gave the same x_N^2 in 21 trials, but in 40 similar trials with $x_0 = -D$, the optimal and suboptimal policies did not give the same x_N^2 in any case.

Although these results are by no means conclusive, they tend to support the conjecture that the one-stage suboptimal policy is a fairly good approximation to the optimal policy for the control system under consideration.

The adoption of this suboptimal policy simplifies the control problem considerably.

The approximate analysis of the control system with the one-stage policy will be presented next.

The state variable under the suboptimal policy is given by

$$x_{n+1} = \alpha x_n + r_n + v_n, \tag{60}$$

where

$$v_n = -m\cdot\operatorname{sgn}(\alpha x_n + (2p-1)c),$$

$$r_n = \begin{cases} +c & \text{with probability } p \\ -c & \text{with probability } 1-p \end{cases} \tag{7$'$}$$

$$0 < c < m, \qquad 0 < \alpha < 1.$$

Let us note that if $x_n > 0$, then $x_{n+1} < x_n$, and if $x_n < x_c$, then $x_{n+1} > x_n$ where $x_c = -(2p-1)c/\alpha$, because $(m-c) > 0$ by assumption. Namely, if the system deviation from the origin (the position of the equilibrium in the absence of the random disturbance r) is positive, the deviation at the next stage will invariably be smaller than the present one. If the present deviation is less than x_c, then the system will move towards the origin at the next stage.

For $x_c < x_n < 0$,

$$v_n = -m,$$

and

$$x_{n+1} > \alpha x_c - c - m = -(m + 2pc).$$

That is to say, once x_n becomes greater than $x_L = \min[-(m+2pc),\, -(2p-1)c/\alpha]$, then x_k can not become smaller than x_L, for all $k \geq n$, *i.e.*, x_L is the lower bound. Similarly, one sees that

$$x_u = m + 2c(1-p)$$

provides an upper bound. That is to say, if $x_n \leq x_u$, then $x_k \leq x_u$ for all $k \geq n$. Simulated system behaviors are functions of the sequence of random variable r_n. That is to say, different orderings of $+c$ and $-c$ with n in $\{r_n\}$ will, in general, result in different behavior of x_n with n. Fig. 3 shows three such typical simulated system behaviors with one-stage suboptimal policy.

One noticeable feature of the system behavior is that x_n approaches the origin for a few stages consecutively, and then x_n jumps away from the origin; and, from there, it again approaches the origin for the next few stages.

Let us now investigate the number of stages before a jump occurs. This serves to show how the jerkiness of the system depends on system parameters α, c, m and p.

Suppose that $x_L = -(m+2pc)$, and $x_0 = x_L$. Let k be

the first time that $x_k > x_c$ is realized. The largest k will be realized when r_n is equal to $-c$ for all k stages, and the smallest k will be given when $r_n = +c$ occurs consecutively.

In the former

$$x_k = -\alpha^k(m + 2pc) + \frac{(m - c)(1 - \alpha^k)}{1 - \alpha} \geq -\frac{(2p - 1)c}{\alpha}. \tag{74}$$

In the latter

$$x_k = -\alpha^k(m + 2pc) + \frac{(m + c)(1 + \alpha^k)}{1 - \alpha} \geq -\frac{(2p - 1)c}{\alpha}. \tag{75}$$

From (74)

$$k \geq \frac{\log\left[\frac{m-c}{1-\alpha} + \frac{(2p-1)c}{\alpha}\right] - \log\left[m + 2pc + \frac{m-c}{1-\alpha}\right]}{\log[\alpha]}. \tag{76}$$

From (75)

$$k \geq \frac{\log\left[\frac{m+c}{1-\alpha} + \frac{(2p-1)c}{\alpha}\right] - \log\left[m + 2pc + \frac{m+c}{1-\alpha}\right]}{\log[\alpha]}. \tag{77}$$

With $\alpha = \frac{7}{8}$, $p = 0.625$, $c = \frac{1}{16}$ and $m = 9/128$, $k \geq 7$ in (76) and $k \geq 1$ in (77). Therefore, k may be expected to range from 1 to 8; $k = 8$ is realized with probability $(1-p)^8$, and $k = 1$ is realized with probability p.

If $x_0 = x_u$, then the similar argument gives

$$k \geq \frac{\log\left[\frac{m \mp c}{1 - \alpha} - \frac{(2p - 1)c}{\alpha}\right] - \log\left[m + 2c(1 - p) + \frac{m \mp c}{1 - \alpha}\right]}{\log[\alpha]}. \tag{78}$$

The maximum k is given with $-$ sign, and the minimum k is given with $+$ sign in (78). The range of k is from 1 to 11. The maximum k occurs with probability p^k, and the minimum k with probability $(1-p)$.

Thus, after n becomes such that $x_L \leq x_n \leq x_u$ is realized, then the bounds on the number of consecutive stages k where the system approaches the origin monotonically are obtained. Since Fig. 3 shows x_n starting from $x_0 = D$, and not from $x_0 = x_L$ or x_u, the computed range of k need not be a good estimate for the range of k in this figure. As a matter of fact, the range of k in Fig. 3 is expected to be narrower, and this is indeed the case as seen from Fig. 3.

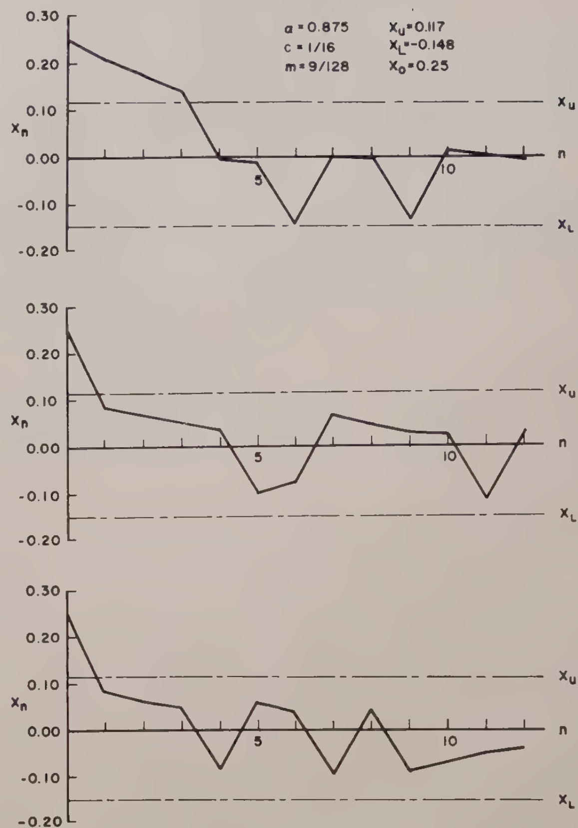

Fig. 3—Typical behaviors of the state variable x_n, with the numbers of past control actions n, for $p = 0.625$.

VI. Conclusions

It was shown that for a class of final value systems considered in this paper, $k_n(x, z)$ is concave in the parameter z which represents the *a priori* knowledge on the states of nature.

The structural dependence of $k_n(x, z)$, when $\phi(x_N)$ is quadratic in x_N, was investigated in detail under the integral constraint on the control variable v, and it was shown that $k_n(x, z) = w_n(z)x^2$ and that $w_n(z)$ is quadratic in the neighborhood of $z = 1$ or 0. The optimal control variable was shown to be monotonic in z, and linear in x.

When the control variable is of the binary form $\pm m$, no explicit dependence of $k_n(x, z)$ and $v_n(x, z)$ on x is obtained.

However, one-stage suboptimal policy was shown to be a fairly good approximation to the optimal policy and was used to establish lower and upper bounds on the system deviations.

Appendix I

The Concavity of $k_n(x, z)$ in z

From (10), $k_1(x, z)$ is linear in z (special case of concavity). Assume that $k_n(x, z)$ is concave in z. Then,

$$k_n(x^+, z') \geq z'k_n(x^+, 1) + (1 - z')k_n(x^+, 0),$$
$$k_n(x^-, z') \geq z'k_n(x^-, 1) + (1 - z')k_n(x^-, 0). \quad (79)$$

Also from (15),

$$k_{n+1}(x, z) \geq \min_v [zp_1 + (1 - z)p_2][z'k_n(x^+, 1) + (1 - z')k_n(x^+, 0)] + [z(1 - p_1) + (1 - z)(1 - p_2)] \cdot [z''k_n(x^-, 1) + (1 - z'')k_n(x^-, 0)]. \quad (80)$$

Since from (11) and (12),

$$[zp_1 + (1 - z)p_2]z' = zp_1,$$
$$[z(1 - p_1) + (1 - z)(1 - p_2)]z'' = z(1 - p_1),$$

using the relation

$$\min_v [f(v) + g(v)] \geq \min_v f(v) + \min_v g(v)$$

where $f(v) \geq 0$, $g(v) \geq 0$, (79) becomes

$$\begin{aligned} k_{n+1}(x, z) &\geq \min_v [zp_1k_n(x^+, 1) + (1 - z)p_2k_n(x^+, 0) + z(1 - p_1)k_n(x^-, 1) + (1 - z)(1 - p_2)k_n(x^-, 0)], \\ &\geq z \min_v [p_1k_n(x^+, 1) + (1 - p_1)k_n(x^-, 1)] + (1 - z) \min_v [p_2k_n(x^+, 0) + (1 - p_2)k_n(x^-, 0)] \\ &= zk_{n+1}(x, 1) + (1 - z)k_{n+1}(x, 0), \end{aligned} \quad (81)$$

i.e.,

$$k_{n+1}(x, z) \geq zk_{n+1}(x, 1) + (1 - z)k_{n+1}(x, 0). \quad (82)$$

Therefore, $k_{n+1}(x, z)$ is concave in z. This completes the mathematical induction on n.

Appendix II

The Proof of $h_n(x; p) = h_n(-x; 1-p)$

The relation (82) is assumed for all k.

$$v_k(x; p) = - v_k(-x; 1 - p). \quad (83)$$

Then

$$h_k(x; p) = h_k(-x; 1 - p) \quad (84)$$

is true for all k. This can be seen as follows:

For $k=1$, from (63′),

$$h_1(x; p) = 4p(1 - p)c^2 + [\alpha x + (2p - 1)c + v_1(x; p)]^2$$
$$\begin{aligned} h_1(-x; 1 - p) &= 4p(1 - p)c^2 + [-\alpha x + (2(1 - p) - 1)c + v_1(-x; 1 - p)]^2 \\ &= 4p(1 - p)c^2 + [\alpha x + (2p - 1)c + v_1(x; p)]^2. \end{aligned}$$

Therefore, (84) is true for $k=1$.

Assume that it is true for all k less than n. Then for $k=n$ from (9)

$$h_n(x; p) = [ph_{n-1}(x^+; p) + (1 - p)h_{n-1}(x^-; p)]_{v_n(x; p)}, \quad (85)$$
$$h_n(-x; 1 - p) = [(1 - p)h_{n-1}((-x)^+; 1 - p) + ph_{n-1}((-x)^-; 1 - p)]_{v_n(-x; 1-p)}, \quad (86)$$

where

$$x^+ = \alpha x + c + v_n(x; p), \qquad x^- = \alpha x - c + v_n(x; p),$$
$$\begin{aligned} (-x)^+ &= \alpha(-x) + c + v_n(-x; 1 - p) = - (x^-) \\ &\quad + v_n(-x; 1 - p) + v_n(x; p) = - (x^-) \end{aligned}$$
$$(-x)^- = \alpha(-x) - c + v_n(-x; 1 - p) = - (x^+).$$

Now, (86) can be rewritten as

$$\begin{aligned} h_n(-x; 1 - p) &= [(1 - p)h_{n-1}(-(-x)^+; p) + ph_{n-1}(-(-x)^-; p)]_{v_n(-x; 1-p)} \\ &= [(1 - p)h_{n-1}(x^-; p) + ph_{n-1}(x^+; p)]_{v_n(x; p)} \\ &= h_n(x; p). \end{aligned} \quad (87)$$

This completes the mathematical induction loop on k. The truth of (83) can be proved in the following manner.

For $k=1$, from (64)

$$\begin{aligned} v_1(x; p) &= - m \cdot \operatorname{sgn} (\alpha x + (2p - 1)c) \\ v_1(-x; 1 - p) &= - m \cdot \operatorname{sgn} (-\alpha x - (2p - 1)c) \\ &= - v_1(x; p). \end{aligned} \quad (88)$$

Assume that (83) is true for all indexes up to k. Then

$$\begin{aligned} v_{k+1}(x; p) &= - m \cdot \operatorname{sgn} (\alpha^{k+1}x + (2p - 1)c(1 + \alpha + \cdots + \alpha^k) + \gamma_1 + \cdots + \alpha^{k-1}\gamma_k), \\ v_{k+1}(-x; 1 - p) &= - m \cdot \operatorname{sgn}[-\alpha^{k+1}x - (2p - 1)c(1 + \alpha + \cdots + \alpha^k) + \tilde{\gamma}_1 + \cdots + \alpha^{k-1}\tilde{\gamma}_k], \end{aligned} \quad (89)$$

where $\tilde{\gamma}_i$ is γ_i for the $(-x, 1-p)$ pair.

Since γ_i is equal to $v_i(x; p)$ or equal to $(2p-1)v_i^+$ as defined in (70),

$$\tilde{\gamma}_i = \begin{cases} v_i(-x; 1 - p) = - v_i(x; p) \text{ or} \\ -(2p - 1)v_i^+(-x; 1 - p) \end{cases}, \quad (90)$$

and therefore, $\tilde{\gamma}_i = -\gamma_i$, $i=1, 2, \cdots, k$, since $v_i^+(-x; 1-p) = -v_i^+(x; p)$ is true by assumption for $i=1, 2, \cdots, k$.[9] This completes the loop of mathematical induction on the index k.

Appendix III

Computational Procedures

Let us consider a set C consisting of a finite number of x values from the domain of x, $|x| \leq D$. Let us denote elements of C by $c_1, c_2, \cdots, c_m$.

[9] In the induction, it is assumed that $v_n(x; p) = -v_n(-x; 1-p)$ holds for $n=1, 2, \cdots, k$. Hence, $v_n^+(x; p) = -v_n^+(-x; 1-p)$ also holds for $n=1, 2, \cdots, k$.

The functional value of $h_1(x)$ is evaluated at each of $x=c_i, i=1, 2, \cdots, m$. This can be done because explicit value for x_+ and x_- are given by (62) for each of $x=c_i$ as functions of v.

The functional value of $h_2(x)$ at $x=c_1$ is evaluated next. It is seen from (9′) that to compute $h_2(c_i)$, it is necessary to have $h_1(c_{1+})$ and $h_1(c_{1-})$, where

$$c_{1\pm} = \alpha c_1 \pm c + v.$$

Since in general $c_{1\pm}$ do not belong to the set C, it is necessary to approximate $h_1(c_{1\pm})$ from a set of functional values $h_1(c_i)$, $i=1, 2, \cdots m$.

The same process is repeated for $x=c_i$, $i=2, \cdots, m$ to obtain $h_2(x)$ at each of the element of C. It is now obvious how to continue to obtain $h_n(x)$, $n=3, 4, \cdots$. Generally speaking, the larger the value of m, the better the accuracy of computation and the longer it takes to compute the solutions.

Acknowledgment

The author is indebted to Prof. G. Estrin of the University of California at Los Angeles, who first suggested the study of dynamic programming and was very helpful in numerous ways in the course of the investigation.

The author would also like to express his gratitude to Dr. R. Bellman, Dr. R. Kalaba of the RAND Corporation, and to Prof. G. W. Brown of the University of California for their many helpful discussions.

Bibliography

[1] R. Bellman and J. M. Danskin, Jr., "A Survey of the Mathematical Theory of Time-Lag, Retarded Control and Hereditary Processes," RAND Corp., Santa Monica, Calif., Rept. No. R-256; March, 1954.
[2] R. Bellman and R. Kalaba, "Dynamic programming and adaptive processes: mathematical foundation," IRE Trans. on Automatic Control, vol. AC-5, pp. 5–10; January, 1960.
[3] R. Bellman, "On the application of the theory of dynamic programming to the study of control processes," *Proc. Symp. on Nonlinear Circuit Analysis*, Polytechnic Inst. of Brooklyn, Brooklyn, N. Y., pp. 199–213; April, 1956.
[4] R. Bellman, "Dynamic programming and stochastic control processes," *Information and Control*, vol. 1, pp. 228–239; September, 1958.
[5] R. E. Kalman and R. W. Koepcke, "Optimal syntheses of linear sampling control systems using generalized performance indexes," *Trans. ASME*, vol. 80, pp. 1820–1826; November, 1958.
[6] M. Aoki, "Dynamic Programming and Numerical Experimentation as Applied to Adaptive Control Processes," Ph.D. dissertation, Dept. of Engrg., University of California, Los Angeles; November, 1959.
[7] W. Feller, "An Introduction to Probability Theory and Its Applications," John Wiley and Sons, Inc., New York, N. Y., vol. 1; 1957.
[8] C. W. Steeg and M. V. Mathews, "Final-value control synthesis," IRE Trans on Automatic Control, vol. AC-2, pp. 6–16; February, 1957.
[9] R. C. Booton, Jr., "Optimum design of final-value control systems," *Proc. Symp. on Nonlinear Circuit Analysis*, Polytechnic Inst. of Brooklyn, Brooklyn, N. Y., pp. 233–241; April, 1956.
[10] R. C. Booton, Jr., "Final-value systems with Gaussian inputs," IRE Trans. on Information Theory, vol. IT-2, pp. 173–175; September, 1956.
[11] R. Bellman, "Dynamic Programming," Princeton University Press, Princeton, N. J.; 1957.
[12] D. Blackwell and M. A. Girshick, "Theory of Games and Statistical Decisions," John Wiley and Sons, Inc., New York, N. Y.; 1954.
[13] R. Bellman, "Dynamic programming and Lagrange multipliers," *Proc. Natl. Acad. Sciences*, vol. 42, pp. 767–769; October, 1956.
[14] R. Bellman, "Functional equations in the theory of dynamic programming, VI: A direct convergence proof," *Ann. Math.*, vol. 65; pp. 215–233; March, 1957.
[15] R. E. Beckwith, "Analytic and Computational Aspects of Dynamic Programming Process of High Dimensions," Ph.D. dissertation, Dept. of Math., Purdue Univ., Lafayette, Ind.; June, 1959.
[16] H. A. Meyer, "Symposium on Monte Carlo Methods," John Wiley and Sons, Inc., New York, N. Y.; 1956.
[17] H. Osborn, "The problem of continuous programs," *Pacific J. Math.*, vol. 6, pp. 721–731; Winter, 1956.
[18] R. E. Bellman, I. Glicksburg, and O. Gross, "Some Aspects of the Mathematical Theory of Control Processes," RAND Corp., Santa Monica, Calif., Rept. No. R-313; January, 1958.

Discussion

P. M. De Russo (Rensselaer Polytechnic Inst., Troy, N. Y.). The techniques of this paper are valuable, but they generally suffer from the fact that the systems which result are extremely complicated, and the practicality of their physical realization is therefore often doubtful. Dr. Aoki says, "The suboptimal policy simplifies the control problem considerably and is a fairly good approximation to the optimal policy for the control system under consideration." But is this not saying that, in many situations, the conventional techniques which have been utilized for some time are more desirable when practical physical realization is included in the specifications? It appears that the value of the techniques of this paper are that the "best" system is designed and can be utilized as a standard for evaluating nonoptimum systems and for investigating the ultimate limitations to system performance. Also, I would like Dr. Aoki's comments on the convergence of the discrete approximation in the continuous case.

Author's Comments: Dr. DeRusso pointed out correctly one of the major uses of the techniques discussed in this paper, that is to say, the solutions of functional equations of dynamic programming give optimal solutions which serve as standards for evaluating nonoptimal solutions.

There exist, however, at least two other major advantages of the techniques discussed in the paper over the more conventional techniques.

One is the fact that dynamic programming approach furnishes not only optimal solutions, but also structural information on optimal solutions, which in turn may be used to advantage in devising "good" suboptimal solutions. It is to be noted that ordinary numerical procedures will not give structural information on optimal solutions. Furthermore, the fact that dynamic programming admits a new kind of approximation, *approximation in policy space*, which is a more natural approximation in many realistic situations, cannot be ignored.

Another point is the fact that, in spite of many restrictive assumptions, the formulation of adaptive processes via dynamic programming, as done by Bellman and Kalaba, is the only general mathematical formulation of stochastic and adaptive control processes.

In my paper, computational solutions are considered

only for sampling control systems, and only analytic approximations are discussed for the continuous control system of Section III.

However, when the process is continuous, and it is desired to solve functional equations of dynamic programming technique numerically, it is necessary to use difference equations instead of differential equations. This is always the problem we are faced with in trying to use a digital computer in obtaining solutions to problems which do not admit analytic solutions.

The question of convergence of solutions of difference equations to those of corresponding differential equations naturally arises. There are several papers [11], [14], [15], [17] in which this question is investigated. The results, however, are still confined to rather specialized type of functions.

Although the dynamic programming technique is a powerful tool for dealing with multi-stage decision processes, of which control processes are one special kind, actual solutions of functional equations are not always routine matters, especially when the dimension of problems exceeds three or four. Computational solutions of problems of high dimensions are at present limited by capabilities of available digital computers, and by various analytic difficulties. This situation will be remedied as soon as more powerful digital computers become available, and when more attention has been focused upon the problem.

Mathematical Aspects of the Synthesis of Linear Minimum Reponse-Time Controllers*

E. BRUCE LEE†

Summary—**A procedure is presented for finding, in a systematic matter, the forcing function to be applied to a process to give time-optimal control. The results are limited to the case where the process to be controlled has dynamics which are completely known and are adequately described by a system of linear differential equations. It has also been assumed that the process is not subject to unknown disturbances.**

The implication of the method presented is that any process as limited above can be time-optimally controlled, in those cases where a time-optimal controller exists, provided an underlying system of transcendental equations can be solved. This system of transcendental equations can be quite easily solved for systems of order less than three, even with two forcing functions, and for various other special cases of higher order. The results of a study which indicate that the method can be applied in quite general situations are being presented elsewhere [1].

Historical Background

AROUND 1950 a number of people became interested in the control improvement that could be achieved by the use of nonlinear feedback. One of these nonlinearities was the relay, where it was established, on the basis of plausible arguments and experimentation, that the relay controller, if properly switched, would give a smaller response time than any other controller subject to the same amplitude limitation. The first results were very restrictive as to what could be controlled, but there were many results, and all were of great engineering significance as evidenced by the results achieved [2]–[16].

One of the first restrictions that had been imposed was to restrict the results to processes which had real distinct characteristic roots. This first restriction was in part removed by Bushaw [17], where he considers second order processes with complex characteristic roots. See also [12] and [27]. Another restriction imposed was that the differential equation describing the process not contain derivatives of the forcing function; *i.e.*, if the plant was described by means of a transfer function, the transfer function could not have any numerator dynamics. This problem was discussed by Smith [19], where he introduced a new coordinate to handle the jump involved when the relay switches. Hung and Chang also considered this problem [18]. A little survey of the literature shows that this problem had been encountered in the statistical design of control systems. The general transformation to eliminate numerator dynamics can be found in [20]; this result can also be found in Stone, *et al.*[1]

One problem inherent in the design of time-optimal controllers, as well as in all control concepts that control the state vector, is that all elements of the state vector be measurable. For processes of order two or three, it may be possible to generate the state vector by

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[1] See [27], p. II-7.

successive differentiation of a measured variable. In practice, it is hard to obtain derivatives of higher than two or three. This problem was attacked by Harvey,[2] where he shows that most systems of high order are basically coupled second order systems and that, in most cases, there does exist a transformation to measurable coordinates.

One of the first extensions of the concept of relay control to systems of a quite general nature was made by Bellman, Glicksberg and Gross [23] or [24], where they give a proof of the previously accepted theorem which states that relay control is time-optimal. Their developments were restricted to processes with real distinct characteristic roots and to processes where the control matrix is assumed to be non-singular.

Bass [22] presented without proof several important theorems connected with the optimum relay controller. He was the first to indicate that the differential equation adjoint could be used in determining the zeros of the forcing function. LaSalle [25], [26] has also presented a number of important theorems related to time-optimal control. Work in Russia, primarily influenced by the fine leadership of Pontriagin, has established quite general results in the area of time-optimal control. Russian results contain many important general items connected with the existence and uniqueness of optimal controllers. Of great general significance is Pontriagin's principle of the maximum. See [28]–[33] for a complete account of the Russian work. The problem of time-optimal control has also been studied by Desoer [34] using classical calculus-of-variation techniques and by Krasovskii [35]–[38]. Also of a related nature is the paper by Rozonoer [39], which discusses the use of Pontriagin's maximum principle and its equivalence in many cases to Bellman's dynamic programming [24].

In all this work, the most important question of engineering interest is: when does one switch the relay sign? In most of the results it is concluded that the solution to the adjoint differential equation will provide the zeros of the relays. The results of this paper show that the basic structure of the switching equation is connected with a set of simultaneous transcendental equations, and that to find the relay switchings one must in some manner solve these equations.

Statement of the Problem

The process control problem of concern here is to select an allowable control vector $\bar{u}(t)$ (allowable as defined below) to carry the process, as represented by the state vector $\bar{x}(t)$ from an initial state ξ_0 to a prescribed state, $\xi(t)$, in minimum time.

If $\bar{u}(t)$, the forcing vector, is restricted to a closed region, $\bar{\Omega}$, where specifically $|\bar{u}_i| \leq m_i$, $i=1, 2, \cdots, r$, then the control vector is said to be allowable.

[2] *Ibid.*, ch. 2.

The process under control to be considered here is assumed to be governed by the linear vector differential equation

$$\frac{d\bar{x}}{dt} = A(t)\bar{x}(t) + B(t)\bar{u}(t), \tag{1}$$

which has the solution [40]

$$\bar{x}(t) = \Phi(t)\xi_0 + \int_0^t \Phi(t)\Phi^{-1}(t')\bar{\omega}(t')dt', \tag{2}$$

where $\bar{\xi}_0$ is the initial condition vector, $\bar{x}(o)$, and $\bar{\omega}(t) = B\bar{u}(t')$ is the forcing vector.[3] Note that if A is a constant matrix,

$$\Phi^{-1}(t) = \Phi(-t), \tag{3}$$

and (2) can be written

$$\bar{x}(t) = \Phi(t)\bar{\xi}_0 + \int_0^t \Phi(t-t')\bar{\omega}(t')dt'. \tag{4}$$

Because it is required that $\bar{x}(t) = \bar{\xi}(t)$ at some future time, $t = t_r$, and that the time interval $(0, t_r)$ is to be a minimum, a requirement is that

$$\Phi^{-1}(t_r)\xi(t_r) - \xi_0 = \int_0^{t_r} \Phi^{-1}(t')\bar{\omega}(t')dt', \tag{5}$$

for some $\bar{\omega}$. Thus, the aim is to find the smallest $t_r \geq 0$ for which an allowable control vector $\bar{\omega}$ is, under the linear mapping given by (5), taken into the vector $\Phi^{-1}(t_r)\bar{\xi}(t_r) - \bar{\xi}_0$.

The following two theorems concerning minimum-response-time control can be easily established for the process governed by the differential equation (1):

Theorem I—If there exists a time optimal controller for the process (1), with $\bar{u} \in \bar{\Omega}$, then there exists a time optimal relay controller, *i.e.*, the control function can be on the boundary of $\bar{\Omega}$ for all t in the time interval $(0, t_r)$. (A simple extension of the proof presented by Bellman, Glicksberg and Gross [23] will establish this theorem.)

Theorem II—If $\bar{x}(0)$ is in the region for which there exists a time optimal controller, and if the n characteristic roots of A, as seen in (1), are real and distinct, the components of the time optimal control vector $\bar{u}$ should each change sign, at most, $n-1$ times in the time interval $(0, t_r)$. (A proof of this theorem is also outlined by Bellman, Glicksberg and Gross [23].)

A sufficient condition, if A is a constant matrix, for insuring the existence of a time optimal regulator is

[3] The matrix $\Phi(t)$ is the solution of the matrix system

$$\frac{d\Phi}{dt} = A\Phi, \qquad \Phi(0) = I,$$

where I is the unit matrix. If A is a constant matrix, the matrix Φ is the matrix exponential

$$\Phi = e^{At}.$$

that the matrix A of (1) have characteristic roots with negative real parts. This is not a necessary condition, as is well known [25], for even in the case of unstable characteristic roots there may exist a region $\bar{\chi}$ over which it is possible to regulate optimally. An example of a situation where it is not possible to optimally regulate would be when $S^{-1}B=0$. [See (10).] In this case no matter how much power is available or how intelligently used, the region $\bar{\chi}$ will be empty if any of the characteristic roots of the matrix A have a positive real part.

Theorem I is very important in the synthesis of the minimum-time-response controller, since it enables the indicated integration of (5) to be carried out. The results of the integration are then used in the procedure for finding the optimal control function. The contribution of this paper is to show how it is possible to go from Theorem I to a set of necessary equations from which the optimum control function can be found by means of a control computer. It will also be indicated how the switching equations, in terms of the state variables, can be found for the lower-order processes.

Theorem II is not important to the method presented here, but if the conditions under which it holds are met, it will represent a major simplification of the method.

In the work that follows it will be assumed that the matrices A and B are constant and that A has stable characteristic roots. Also it is assumed that $\bar{\xi}(t)=0$. Neither of these assumptions is necessary in the use of the method presented. (The modifications for the more general case are obvious.)

Single-Degree-of-Freedom Processes

The result presented in this section is actually a special case of the more general result presented later in this paper. From an engineering point of view, it appears that the simpler case should be studied first.

The equation for the single-degree-of-freedom process can be written as

$$\frac{d^n x}{dt^n}+b_1\frac{d^{n-1}x}{dt^{n-1}}+\cdots+b_n x=u,\ b_i=\text{constants},\tag{6}$$

which can be put in the form of (1) by letting

$$x_1 = x$$

and

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ &\vdots\\ \dot{x}_n &= -b_1x_n - b_2x_{n-1}+\cdots+-b_nx_1+u.\end{aligned}\tag{7}$$

Thus

$$\frac{d\bar{x}}{dt}=A\bar{x}+B\bar{u},$$

where

$$A=\begin{bmatrix}0 & 1 & 0\cdots 0 & \\ 0 & 0 & 1 & 0\cdots\\ \vdots & & & \\ -b_n & -b_{n-1} & \cdots & -b_1\end{bmatrix},\quad \bar{x}(t)=\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix},$$

$$B=\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ 0 & & & \\ \vdots & & & 0\\ 0 & 0 & & 1\end{bmatrix},\quad \bar{u}=\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ u\end{bmatrix}.$$

If distinct characteristic roots are assumed, the solution to (6) can be written in terms of principal coordinates, v_i, (v_i is a linear combination of the original coordinates x_i) as

$$\bar{v}(t)=e^{Ct}v_0+\int_0^t e^{C(t-t')}S^{-1}\bar{\omega}(t')dt',\tag{8}$$

where S is $n\times n$ nonsingular matrix such that

$$C=S^{-1}AS=\begin{bmatrix}a_1 & 0 & 0\cdots 0\\ 0 & a_2 & 0\cdots 0\\ \vdots & & \vdots\\ 0 & 0 & \cdots\ a_n\end{bmatrix},$$

and the a_i are the n distinct characteristic roots of the process equation. For optimum response it is required that $\bar{v}(t)=0$ when $t=t_r$, the response time; thus,

$$-\bar{v}_0=\int_0^{t_r}e^{-Ct'}\bar{g}(t')dt',\tag{9}$$

where for a single-degree-of-freedom process as previously hypothesized,

$$\bar{g}(t)=S^{-1}\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ u\end{bmatrix}=\frac{1}{|S|}\begin{bmatrix}c_{11} & c_{12} & \cdots & c_{1n}\\ c_{21} & c_{22} & \cdots & c_{2n}\\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn}\end{bmatrix}\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ u\end{bmatrix}$$

or

$$g_1(t)=\frac{c_{1n}}{|S|}u,\ g_2(t)=\frac{c_{2n}}{|S|}u,\cdots,g_n(t)=\frac{c_{nn}}{|S|}u.\tag{10}$$

By Theorem I, $|ui|=m_i$ for all t in the interval $(0, t_r)$, and it will be assumed on the basis of Theorem II, that a finite but unknown number of control function switchings will include the time optimal controller. (In some cases it is possible to restrict the range of initial condi-

tions to require at most $n-1$ switchings.) The integration of (9) can now be performed, as follows:

$$\bar{v}_m(0) = \int_0^{t_r} e^{a_m t'} g_m(t') dt' = \int_0^{t_1} g_m e^{a_m t'} dt'$$

$$- \int_{t_1}^{t_2} g_m e^{a_m t'} dt + \cdots + \int_{t_k}^{t_r} g_m e^{a_m t'} dt'$$

$$= \frac{g_m}{a_m} [1 - 2e^{a_m t_1} + 2e^{a_m t_2} + \cdots \pm 2e^{a_m t_k} \mp e^{a_m t_r}];$$

$$0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq t_r;$$

$$m = 1, 2 \cdots n; \; k > 0. \quad (11)$$

The first question to be answered is: what is the smallest $t_r \geq 0$ for which the n equations of (11) are satisfied? By answering this question, the present relay sign is automatically determined, *i.e.*, it will be the same as the sign preceding the first term, $2e^{a_m t_i}$, which has a nonzero t_i. To get at the minimization of t_r, one equation of (11) is solved for t_r, in terms of the other switching times, t_i,

$$t_r = F(t_1, t_2, \cdots t_k), \quad (12)$$

and t_r is eliminated from the remaining $n-1$ equations to obtain $n-1$ constraint equations,

$$G_i = G_i(t_1, t_2, \cdots t_k) = 0, \qquad i = 1, 2, \cdots, n-1.$$

In addition we must have the inequalities

$$0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq t_r. \quad (13)$$

The minimizing procedure is then to look for the minimum of $F(t_1, \cdots, t_k)$ over the set, as given by $G_i(t_1, t_2, \cdots, t_k)$, where the t_i's are restricted by $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq t_r$. To do this and not be committed as to the independent variable, a method developed by Lagrange will be used.

First form the function

$$f = F(t_1, t_2, \cdots, t_k) + \lambda_1 G_1(t_1, t_2, \cdots, t_k) + \cdots + \lambda_{n-1} G_{n-1}(t_1, t_2, \cdots, t_k), \quad (14)$$

where the λ_i are constants, as yet undetermined in value. Treat $t_1, t_2, \cdots, t_k$ as independent variables and write down the necessary condition for an extremal,

$$\frac{\partial f}{\partial t_i} = 0, \qquad i = 1, 2, \cdots, k, \quad (15)$$

subject to the constraints $G_i(t_1, t_2, \cdots, t_k) = 0$, $i = 1, 2, \cdots, n-1$, and where the t_i's are restricted as follows:

$$0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq t_r. \quad (16)$$

It is now obvious how to obtain the proper relay sign. Simply mechanize (13) and (15) to compute the minimizing set of t_i's, and then select the present relay sign as previously indicated. These equations can be solved on-line, with either analog or digital computer type equipment [1]. Note that if Theorem II is applicable, the set of switching times to be considered has, at most, n elements, *i.e.*, $n = k$. Note also that information in addition to the present relay sign will be obtained by this procedure, *e.g.*, a prediction of when the next relay switch will occur. Finally, note that the procedure taken by the Russian authors [28]–[33], [35]–[38] and by Desoer [34] requires the information as presented here to obtain the optimal forcing function; *i.e.*, if given the initial conditions of the adjoint equation, they can obtain a function which has zeros at the switching times, or vice versa. Their equations can be solved for either the switching times or the initial conditions of the adjoint equation, but not for both.

General Linear Process

The general linear process is governed by (1), which has a solution as given by (2). Consider the necessary equation (5) which must occur at $t = t_r$, the response time.

Call the elements of the matrix $\Phi^{-1}(t)$, α_{ij}, *i.e.*,

$$\Phi^{-1}(t) = \begin{bmatrix} \alpha_{11}(t) & \alpha_{12}(t) & \cdots & \alpha_{1n}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) & \cdots & \alpha_{2n}(t) \\ \vdots & & & \\ \alpha_{n1}(t) & \alpha_{n2}(t) & \cdots & \alpha_{nn}(t) \end{bmatrix}, \quad (17)$$

and write

$$\bar{\omega}(t) = \begin{bmatrix} b_{11}u_1 + b_{12}u_2 + \cdots + b_{1m}u_m \\ b_{21}u_1 + b_{22}u_2 + \cdots + b_{2m}u_m \\ \vdots \\ b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nm}u_m \end{bmatrix}, \quad (18)$$

where each $|u_i| = m_i$ for all t in the interval $(0, t_r)$. The integration of (5) is now performed in parts, assuming k_i switching times for each independent forcing function, u_i.

$$-x_1(0) = \int_0^{t_r} \beta_{11}(t')u_1(t')dt' + \int_0^{t_r} \beta_{12}(t')u_2(t')dt' + \cdots + \int_0^{t_r} \beta_{1m}(t')u_m(t')dt',$$

$$-x_2(0) = \int_0^{t_r} \beta_{21}(t')u_1(t')dt' + \int_0^{t_r} \beta_{22}(t')u_2(t')dt' + \cdots + \int_0^{t_r} \beta_{2m}(t')u_m(t')dt',$$

$$\vdots$$

$$-x_n(0) = \int_0^{t_r} \beta_{n1}(t')u_1(t')dt' + \int_0^{t_r} \beta_{n2}(t')u_2(t')dt' + \cdots + \int_0^{t_r} \beta_{nm}(t')u_m(t')dt'. \quad (19)$$

Carry out the integration just for the first equation, letting $t_{11}, t_{12}, t_{13}, \cdots, t_{1k_1}$ be the k_1 switching times for u_1; $t_{21}, t_{22}, \cdots, t_{2k_2}, \cdots, u_2; \cdots, u_n$:

$$
\begin{aligned}
-x_1(0) = {} & m_1\left[\int_0^{t_{11}}\beta_{11}(t')dt' - \int_{t_{11}}^{t_{12}}\beta_{11}(t')dt' + \cdots \int_{t_{1k1}}^{t_r}\beta_{11}(t')dt'\right] \\
& + m_2\left[\int_0^{t_{21}}\beta_{12}(t')dt' - \int_{t_{21}}^{t_{22}}\beta_{12}(t')dt' + \cdots \int_{t_{2k2}}^{t_r}\beta_{12}(t')dt'\right] \\
& \vdots \\
& + m_m\left[\int_0^{t_{m1}}\beta_{1m}(t')dt' - \int_{t_{m1}}^{t_{m2}}\beta_{1m}(t')dt' + \cdots \int_{t_{mk_{m'}}}^{t_r}{}_{1m}(t')dt'\right],
\end{aligned}
\tag{20}
$$

with similar equations for $x_2(0) \cdots x_n(0)$. Now solve one of the above equations, as before, for t_r:

$$
t_r = F(t_{11}, t_{12} \cdots t_{1k_1}, t_{21} \cdots \cdots t_{mk_m}), \tag{21}
$$

and eliminate t_r from the remaining $n-1$ equations to obtain $n-1$ constraint equations,

$$
G_i = G_i(t_{11}, t_{12} \cdots t_{mk_m}); \qquad i = 1, 2, \cdots, n-1. \tag{22}
$$

To find the minimizing set of switching times, proceed as before, forming the function.

$$
f = F + \lambda_1 G_1 + \lambda_2 G_2 + \cdots + \lambda_{n-1}G_{n-1}, \tag{23}
$$

from which we calculate

$$
\frac{\partial f}{\partial t_{ij}} = 0; \quad \begin{matrix} i = 1, 2, \cdots, m; \\ m' = 1, 2, \cdots, m \end{matrix} \quad = 1, 2, \cdots, km', \tag{24}
$$

subject to the constraints G_i, $i=1, 2 \cdots n-1$ and where the t_{ij}'s are restricted as follows:

$$
0 \le t_{i1} \le t_{i2} \le \cdots \le t_r, \quad i = 1, 2, \cdots, m.
$$

The sign of each relay is then chosen to correspond to the appropriate term involving the first nonvanishing t_{ij}. For more general results and the use of other optimum criteria see [41].

A Second-Order Example

Consider a particular second-order process as described by the differential equation,

$$
\begin{aligned}
\dot{x}_1 &= a_{12}x_2 + b_{11}u_1 \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + b_{22}u_2.
\end{aligned}
\tag{25}
$$

Assume that this equation has real distinct characteristic roots $-a$, $-b$. The fundamental matrix $\Phi(t)$ is then

$$
\Phi(t) = \begin{bmatrix} \dfrac{be^{-at} - ae^{-bt}}{b-a} & \dfrac{e^{-at} - e^{-bt}}{b-a} \\[2ex] \dfrac{-abe^{-at} + abe^{-bt}}{b-a} & \dfrac{-ae^{-at} + be^{-bt}}{b-a} \end{bmatrix}. \tag{26}
$$

Because A was assumed to be a constant matrix.

$$
\Phi^{-1}(t) = \Phi(-t) = \begin{bmatrix} \dfrac{be^{at} - ae^{bt}}{b-a} & \dfrac{e^{at} - e^{bt}}{b-a} \\[2ex] \dfrac{-abe^{at} + abe^{bt}}{b-a} & \dfrac{-ae^{at} + be^{bt}}{b-a} \end{bmatrix}. \tag{27}
$$

Writing out (5) for the regulator problem,

$$
\begin{aligned}
x_1(0) &= \int_0^{t_r}\left[\frac{be^{at'} - ae^{bt'}}{b-a}b_{11}u_1 + \frac{e^{at'} - e^{bt'}}{b-a}b_{22}u_2\right]dt', \\
x_2(0) &= \int_0^{t_r}\left[\frac{-abe^{at'} + abe^{bt'}}{b-a}b_{11}u_1 \right. \\
&\qquad \left. + \frac{-ae^{at'} + be^{bt'}}{b-a}b_{22}u_2\right]dt',
\end{aligned}
\tag{28}
$$

which can, in turn, be written in principal coordinates

$$
\begin{aligned}
v_1 &= ax_1(0) + x_2(0) = \int_0^{t_r} e^{bt'}(ab_{11}u_1 + b_{22}u_2)dt', \\
v_2 &= bx_1(0) + x_2(0) = \int_0^{t_r} e^{at'}(bb_{11}u_1 + b_{22}u_2)dt'.
\end{aligned}
\tag{29}
$$

Because each $|u_i| = 1$ (letting the matrix B carry the magnitude) for all t in the interval $(0, t_r)$, and the characteristic roots $-a$, $-b$ are assumed real and distinct, the above integration can be carried out in $n-1$ parts, assuming an initial sign for the forcing terms, u_1 and u_2.

$$
\begin{aligned}
v_1 &= \frac{-ab_{11}}{b}[1 - 2e^{bt_1} + e^{bt_r}] - \frac{b_{22}}{b}[1 - 2e^{bt_2} + e^{bt_r}], \\
v_2 &= \frac{-bb_{11}}{a}[1 - 2e^{at_1} + e^{at_r}] - \frac{b_{22}}{a}[1 - 2e^{at_2} + e^{at_r}].
\end{aligned}
\tag{30}
$$

Solving for t_r

$$
\begin{aligned}
t_r &= \frac{1}{a}\log\left[-1 + \frac{-v_2 - \left[\dfrac{2bb_{11}}{a}\right]e^{at_1} - \left[\dfrac{2b_{22}}{a}\right]e^{at_2}}{\left[\dfrac{b}{a}b_{11} + \dfrac{b}{a}\right]}\right], \\
t_r &= \frac{1}{b}\log\left[-1 + \frac{-v_1 - \left[\dfrac{2ab_{11}}{b}\right]e^{bt_1}\left[\dfrac{2b_{22}}{b}\right]e^{bt_2}}{\left[\dfrac{ab_{11}}{b} + \dfrac{b_{22}}{b}\right]}\right].
\end{aligned}
\tag{31}
$$

Call the first equation of (31), $F(t_1, t_2)$, and eliminate t_r from the second equation to find the constraint equation

$$
\begin{aligned}
G(t_1, t_2) &= 0 = \left[-1 + \frac{-v_2 - \left[\frac{2bb_{11}}{a}\right] e^{at_1} - \left[\frac{2b_{22}}{a}\right] e^{at_2}}{\frac{bb_{11}}{a} + \frac{b_{22}}{a}} \right]^{1/a} \\
&\quad - \left[-1 + \frac{-v_1 - \left[\frac{2ab_{11}}{b}\right] e^{bt_1} - \left[\frac{2b_{22}}{b}\right] e^{bt_2}}{\frac{ab_{11}}{b} + \frac{b_{22}}{b}} \right]^{1/b} \\
&= [I]^{1/a} - [II]^{1/b}.
\end{aligned} \tag{32}
$$

The minimizing procedure is then to look for the minimum of $F(t_1, t_2)$ over the set in the first quadrant as given by $G(t_1, t_2)$. Let

$$
\begin{aligned}
f &= F(t_1, t_2) + \lambda G(t_1, t_2) \\
&= \frac{1}{a} \log [I] + \lambda [I]^{1/a} - \lambda [II]^{1/b}.
\end{aligned} \tag{33}
$$

Thus,

$$
\frac{\partial f}{\partial t_1} = \frac{-2bb_{11}e^{at_1}}{bb_{11} + b_{22}} \left[\frac{1}{[I]} + \lambda [I]^{(1/a)-1} \right] - \lambda [[II]^{(1/b)-1}] \left[\frac{-2ab_{11}}{ab_{11} + b_{22}} \right] e^{bt_1} = 0, \tag{34a}
$$

$$
\frac{\partial f}{\partial t_2} = -2b_{22}e^{at_2} \left[\frac{1}{[I]} + \lambda [I]^{(1/a)-1} \right] - \lambda [[II]^{(1/b)-1}] \left[\frac{-2b_{22}}{ab_{11} + b_{22}} \right] e^{bt_2} = 0. \tag{34b}
$$

Eqs. (32) and (34) can be used to solve for the switching times and, in effect, the proper relay signs by means of a control computer.

To elucidate further the importance of the procedure presented, consider now the problem of determining the switching boundaries for the two forcing functions in the particular second-order example being considered. A simple algebraic simplification of (34) leads to the equation expressing the relationship between the switching times which occur at the minimum.

$$
t_1 - t_2 = \log \left[\frac{a}{b} \right]^{1/(a-b)}. \tag{35}
$$

Thus, in all cases, the switching times t_1 and t_2 differ by at most a constant. To plot a curve for the switchings of u_1 and u_2, select $a=1$, $b=2$. First note, in this case, that t_1 is larger than t_2 by, at most, log 2, and that if the aim is to find the switching boundary along which both forcing terms are positive, there will be no switches in the forcing terms, *i.e.*, $t_1 = t_2 = 0$. The second condition leads to

$$
\left[\frac{-2v_1}{b_{11} + b_{22}} + 1 \right] = \left[\frac{-v_2}{2b_{11} + b_{22}} + 1 \right]^2. \tag{36}
$$

A simple test of the direction of movement of a point on this boundary will determine which portion is the required switching boundary; *i.e.*, it should be the portion of the curve along which the point moves toward the null point.

By similar reasoning, the other three boundaries can be determined. Condition 1 has not yet been imposed for the minimum which, for this example, is given by (35) with $a=1$, $b=2$. It has been noted before that t_1 is larger than t_2; thus, to find the switching boundary for u_2, it is just necessary to consider the locus of points that are log 2 units of time away from a switching curve as determined previously. To find the switching equation for u_2, consider (29), which can now provide a relationship between the optimum switching times t_1 and t_2 as given by (35). The result of this substitution leads to the equations,

$$
\begin{aligned}
v_1 &= -\frac{1}{2} b_{11}[-7 + e^{2t_r}] - \frac{b_{22}}{2}[-1 + e^{2t_r}], \\
v_2 &= -2b_{11}[-3 + e^{t_r}] - b_{22}[-1 + e^{t_r}].
\end{aligned} \tag{37}
$$

Eliminating t_r, the response time, obtain the following equation which is part of the u_2 switching boundary:

$$
\left[\frac{7b_{11} + b_{22}}{b_{11} + b_{22}} + \frac{2v_1}{b_{11} + b_{22}} \right] = \left[\frac{6b_{11} + b_{22}}{2b_{11} + b_{22}} + \frac{v_2}{2b_{11} + b_{22}} \right]^2. \tag{38}
$$

Fig. 1 indicates, for the particular second-order example considered, what the complete switching boundaries for u_1 and u_2 look like in the phase plane composed of the principal coordinates v_1 and v_2.

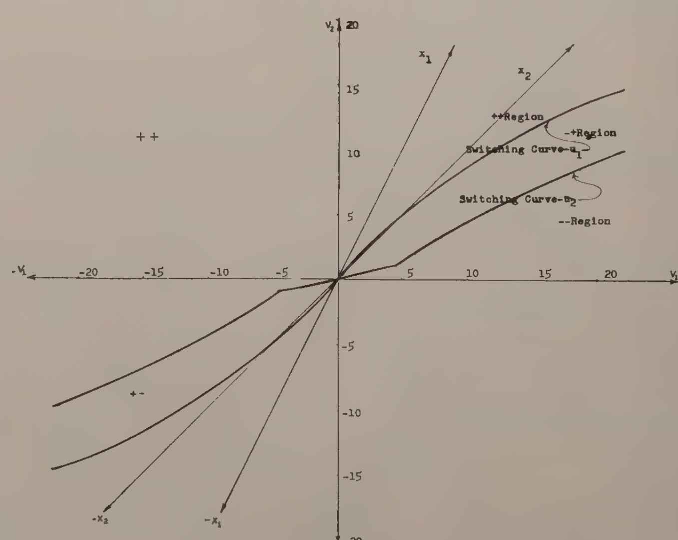

Fig. 1—Optimum switching curves for second order process with two independent forcing terms, $b_{11}=2$, $b_{22}=5$, $a=1$, $b=2$.

Conclusions

A method has been presented which leads in a systematic manner to the desired time optimal forcing function. The method lacks esthetic beauty and does not give any general insight as to the over-all structural properties of the problem, but it does lead to the required engineering solution, which is to know the present sign of each relay forcing term. Admittedly, the method requires a great deal of care in computing the solution to the transcendental equations obtained under the constraints imposed; this difficulty is correctly amplified in the discussion which follows this paper. However, as [1] shows, it is possible with presently available computers to use the method presented here to find, in real time, the time optimal control function for a large class of present-day high-order dynamic processes.

Bibliography

[1] F. Smith, "Control of Higher Order Systems Based on Time Optimal Regulation," presented at IRE Region Conf., Seattle, Wash.; May 24–26, 1960.

[2] D. MacDonald, "Nonlinear techniques for improving servo performance," *Proc. NEC*, vol. 6, pp. 400–421; 1950.

[3] J. B. Lewis, "The use of nonlinear feedback to improve the transient response of a servomechanism," *Trans. AIEE*, vol. 71, pt. II, pp. 449–453; 1952.

[4] R. S. Neiswander and R. H. MacNeal, "Optimization of nonlinear control systems by means of nonlinear feedback," *Trans. AIEE*, vol. 72, pt. II, pp. 262–272; 1953.

[5] I. Flugge-Lotz, "Discontinuous Automatic Control," Princeton University Press, Princeton, N. J.; 1953.

[6] I. Bogner and L. Kazda, "An investigation of the switching criteria for higher order servomechanisms," *Trans. AIEE*, vol. 73, pt. 2, pp. 118–127; 1954.

[7] L. F. Kazda, "Control system optimization using computers as control system elements," *Proc. Computers in Control Systems Conf.*, Atlantic City, N. J., 1957; *Trans. AIEE*, pp. 154–166, 1958.

[8] A. M. Hopkins and M. Iwama, "A study of a predictor-type air-frame controller designed by phase-space analysis," *Trans. AIEE*, vol. 75, pt. II, pp. 1–9; 1956.

[9] A. M. Hopkin, "A phase-plane approach to the compensation of saturating servomechanisms," *Trans. AIEE*, vol. 70, pt. II, pp. 631–639;1951.

[10] A. A. Feldbaum, "Optimum processes in automatic regulations," *Avtomat. i Telemekh.*, vol. 14, no. 5; 1953.

[11] A. A. Feldbaum, "On the synthesis of optimum systems by means of the phase space," *Avtomat. i Telemekh.*, vol. 16, pp. 129–149; 1955.

[12] H. S. Tsien, "Engineering Cybernetics," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 136–159; 1954.

[13] T. M. Stout, "Switching errors in an optimum relay servomechanism," *Proc. NEC*, Chicago, Ill., vol. 9, pp. 188–198; 1953.

[14] T. Doll and T. M. Stout, "Design and analog-computer analysis of an optimum third-order nonlinear servomechanism," *Trans. ASME*, pp. 513–525; 1957.

[15] L. M. Silva, "Predictor servomechanisms," IRE Trans. on Circuit Theory, vol. CT-1, pp. 56–70; March, 1954.

[16] N. J. Rose, "Optimum switching criteria for discontinuous automatic controls," 1956 IRE National Convention Record, pt. 4, pp. 61–67.

[17] D. W. Bushaw, "Contributions to the theory of nonlinear oscillations," in "Optimal Discontinuous Forcing Terms," Princeton University Press, Princeton, N. J., vol. 4, pp. 29–52; 1958.

[18] J. C. Hung and S. S. L. Chang, "Switching discontinuities in phase space," 1957 IRE National Convention Record, pt. 4, pp. 22–26.

[19] O. J. M. Smith, "Feedback Control Systems," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 563–564; 1958.

[20] J. H. Laning and H. B. Battin, "Random Processes in Automatic Control," McGraw-Hill Book Co., Inc., New York, N. Y., p. 191; 1956.

[21] S. Lee, "Optimization of Servomechanisms Having Velocity Saturation," Ph.D. Thesis, University of Minnesota, Minneapolis; 1958.

[22] R. W. Bass, "Equivalent linearization nonlinear circuit synthesis and the stabilization and optimization of control systems," *Proc. Symp. on Nonlinear Circuit Analysis*, Polytechnic Institute of Brooklyn, Brooklyn, N. Y., vol. 6, pp. 163–198; 1956.

[23] R. Bellman, I. Glicksberg, and O. Gross, "On the bang-bang control problem" *Quart. J. Appl. Math.*, vol. 14, pp. 11–18; 1956.

[24] R. Bellman, I. Glicksberg, and O. Gross, "Some Aspects of the Mathematical Theory of Control Processes," The RAND Corp., Santa Monica, Calif., pp. 107–110; 1958.

[25] J. P. LaSalle, "Basic principle of the bang-bang servo," *Bull. Amer. Math. Soc.*, vol. 60, p. 154; 1954.

[26] J. P. LaSalle, "Time optimal control systems," *Proc. Natl. Acad. Sci.*, vol. 45, pp. 573–577; 1959.

[27] R. Stone, C. Johnson, F. Smith, E. B. Lee and A. Harvey, "Time-optimal control of linear systems," Minneapolis-Honeywell, Regulator Co., Minneapolis, Minn., Tech. Rept. No. R-ED 6134; September, 1959.

[28] L. S. Pontriagin, V. G. Boltianskii, and R. B. Gamkrelidze, "On the theory of optimal processes," *Dokl. Akad. Nauk S.S.S.R.*, vol. 110, pp. 7–10; September 1, 1956.

[29] L. S. Pontriagin, V. G. Boltianskii, and R. B. Gamkrelidze, "Optimal control processes," *Uspekhi Matem. Nauk*, vol. 14, pp. 3–20; 1959. (Translation in *Automation Express;* May-June, 1959.)

[30] R. Gamkrelidze, "Theory of time-optimal processes for linear systems," *Izvest. Akad. Nauk S.S.S.R., Seria Matem.*, vol. 22, pp. 449–474; 1958.

[31] R. Gamkrelidze, "On the general theory of optimum processes," *Dokl. Akad. Nauk S.S.S.R.*, vol. 123, pp. 223–226; 1958.

[32] R. Gamkrelidze, "Optimal (regarding quick-response) processes at restricted phase coordinates," *Dokl. Akad. Nauk S.S.S.R.*, vol. 125, pp. 475–478; 1959.

[33] V. G. Boltianskii, "Principle of the maximum in the theory of optimal processes," *Dokl. Akad. Nauk S.S.S.R.*, vol. 119, pp. 1070–1073; 1958.

[34] C. A. Desoer, "The bang-bang problem treated by variational techniques," *Information and Control*, vol. 2, pp. 333–348; 1959.

[35] N. N. Krasovskii, "On the theory of optimum regulation," *Avtomat. i Telemekh.*, vol. 18, pp. 960–970; 1957. (Translation in *Automation and Remote Control;* November, 1957.)

[36] N. N. Krasovskii, "On one particular problem of optimum control," *Priklad. Matem. i Mekhanizatsiya*, vol. 21, pp. 668–669; September, 1957.

[37] N. N. Krasovskii, "To the theory of optimal regulation," *Appl. Math. and Mechanics*, vol. 23, pp. 625–639; 1959.

[38] N. N. Krasovskii, "A contribution to the theory of optimum regulation of nonlinear second order systems," *Dokl. Akad. Nauk S.S.S.R.*, vol. 126, pp. 267–270; 1959.

[39] L. I. Rozonoer, "L. S. Pontriagin's maximum principle in the theory of optimum systems," *Avtomat. i Telemekh.*, vol. 20, pp. 1320–1334; 1959.

[40] E. A. Coddington and N. L. Levinson, "Theory of Ordinary Differential Equations," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 62–78; 1955.

[41] E. B. Lee, "Design of Optimum Multivariable Control Systems," to be published.

Discussion

A. M. Hopkin (University of California, Berkeley): The ideas expressed in this paper are stated in more elegant and general terms, but bear some resemblance to previous work done by the discussor.[1] On the basis of this experience, the following comments seem pertinent.

In (13) of the paper two kinds of constraints are specified, one set up in the form $G_i=0$, and the other in the form $0\leq t_1\leq t_2\leq \cdots \leq t_k\leq t_r$. The first constraint is taken care of by (14) (Lagrange's multipliers), but the

[1] A. M. Hopkin and P. K. C. Wang, "A relay-type feedback control system designed for random inputs," *Trans. AIEE*, vol. 59.

second is not. If a computer is programmed to minimize the Lagrangian function of (14) by setting derivatives with respect to various t_i variables equal to zero, it is possible that a solution might be obtained which did not conform to the second constraint. It is also conceivably possible that for a particular t_i and set of initial conditions there would be no zero of the partial derivative of the Lagrangian function with respect to t_i within the constrained range of t_i, yet the function could be minimum at one end of that range. It is felt that these factors require careful consideration before any control computer is built.

Another possibility which exists is that the Lagrangian function as seen in the multidimensional t_i space might be a surface with many humps and hollows, so that the minimization equations (15) might have many solutions. This event would be likely if some of the characteristic roots of the system equation occurred in complex conjugate pairs.

In this case, the computer would have to search through all possible solution sets to find the particular one which met the constraints and gave a true minimum t_r. Since the computer must solve simultaneous transcendental equations, the total solution might be quite slow in terms of the time scale of the controlled process for any economically designed computer.

The concept of designing a control system on the basis of minimization of the time required to bring the controlled process from an initial state to some desired state was first introduced almost ten years ago. Continuing contributions are being made to the general theory. Unfortunately, it seems that the details involved in the design of actual systems utilizing this theory are troublesome, so that very few practical systems have as yet been built. The main use of the theory so far has been to provide a model for comparison.

Author's Comments: I would like to thank Professor Hopkin for his discussion of my paper. His discussion correctly amplifies a difficulty which is encountered in solving the time optimal control problem. In part, these difficulties have been overcome; the solutions so far obtained are reported by F. Smith [1].

Statistical Design of Digital Control Systems*

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Summary—**This paper is concerned with the problem of statistical design of digital control systems. A simplified design procedure is introduced. The minimum mean-square error is adopted as the performance criterion; and the system design is based upon the Wiener-Kolmogoroff theory of optimum filtering and prediction. The treatment of this optimization problem makes use of the modified *z*-transform technique, and is presented in close parallel with the procedure for the statistical design of linear continuous-data control systems. By making use of some important properties of the auto-correlation sequence, the cross-correlation sequence, the pulse spectral density and the pulse cross-spectral density, the statistical design of digital control systems can be carried out in a simple and systematic manner.**

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Introduction

During the past decade, the analysis and design of digital control systems have been extensively studied. In the past few years, several papers[1–5] have been written on the subject of statistical design of

[1] G. Franklin, "Linear filtering of sampled data," 1955 IRE National Convention Record, pt. 4, pp. 119–128.

[2] R. M. Stewart, "Statistical design and evaluation of filters for the restoration of sampled data," Proc. IRE, vol. 44, pp. 253–257; February, 1956.

[3] A. S. Robinson, "The synthesis of computer-limited sampled-data control systems," *AIEE Special Publication* T-101, pp. 77–78; May, 1958.

[4] S. S. L. Chang, "Statistical design theory for digital-controlled continuous systems," *Trans. AIEE*, vol. 77, pt. 2, pp. 191–201; September, 1958.

[5] J. T. Tou, "Digital and Sampled-Data Control Systems," McGraw-Hill Book Co., Inc., New York, N. Y., Ch. 10; 1959.

sampled-data systems. The procedures proposed in these papers appear to be either involved or incomplete. This paper attempts to study the statistical design problem by an alternate approach so that a simpler design procedure may result. Compact design formulas are derived in close parallel with the statistical design of continuous-data systems.[6-8] Optimum pulse-transfer functions are determined on the basis of the minimization of both the mean-square error and the mean-square sampled error. The design procedures are discussed along with physical realizability and system constraints. In this paper the input signals and the noise of the system are assumed to be stationary random functions.

Some Properties of Correlation Sequence and Pulse Spectral Density

When the minimization of the mean-square error is adopted as the performance criterion, the design of discrete-data systems is usually initiated with the description of the signals by the correlation sequences and pulse-spectral densities.[5] The definitions of auto-correlation sequence, cross-correlation sequence, pulse-spectral density, and cross pulse-spectral density are given in the literature. It has been shown[5] that the correlation sequence and pulse-spectral density possess the following properties:

1) If the response of the pulsed-data system $G(z)$ to an input $x^*(t)$ is $y^*(t)$, then the response of this system to an input $\phi_{xx}(kT)$ is $\phi_{xy}(kT)$, and the response of this system to an input $\phi_{yx}(kT)$ is $\phi_{yy}(kT)$, where $\phi_{xx}(kT)$ and $\phi_{yy}(kT)$ denote the autocorrelation sequence of the pulsed signals $x^*(t)$ and $y^*(t)$, respectively, and $\phi_{xy}(kT)$ and $\phi_{yx}(kT)$ are the cross-correlation sequences between these two pulsed signals.

2) The pulse-spectral densities for the input and the output sequences of a pulsed-data system $G(z)$ are related by

$$\phi_{yy}(z) = G(z)G(z^{-1})\phi_{xx}(z), \tag{1}$$

where $\phi_{xx}(z)$ is the pulse-spectral density for the input sequence $x^*(t)$, and $\phi_{yy}(z)$ is that for the output sequence $y^*(t)$.

3) Assume that two pulsed-data systems $G_1(z)$ and $G_2(z)$ are subjected to input sequences $x_1^*(t)$ and $x_2^*(t)$, respectively, and that the output sequences in response to these inputs are $y_1^*(t)$ and $y_2^*(t)$, respectively. Then the output sequence of the system $G_1(z)$ is $\phi_{x_2y_1}(kT)$ when it is subjected to an input sequence $\phi_{x_2x_1}(kT)$, and the output sequence of the system $G_2(z)$ is $\phi_{y_1y_2}(kT)$ when it is subjected to an input sequence $\phi_{y_1x_2}(kT)$.

4) The cross pulse-spectral densities for the input and the output sequences of two pulsed-data systems $G_1(z)$ and $G_2(z)$ are related by

$$\phi_{y_1y_2}(z) = G_1(z^{-1})G_2(z)\phi_{x_1x_2}(z), \tag{2}$$

where $\phi_{x_1x_2}(z)$ is the cross pulse-spectral density for the input sequences $x_1^*(t)$ and $x_2^*(t)$, and $\phi_{y_1y_2}(z)$ is that for the output sequences $y_1^*(t)$ and $y_2^*(t)$.

Furthermore, it can be shown that the following input-output relationships also hold:

5) If the response of the linear system $G(s)$ to an input sequence $x^*(t)$ is $y(t)$, then the "modified" pulse-spectral density for the output $y(t)$ is related to the pulse-spectral density for the input $x^*(t)$ by

$$\phi_{yy}(z, m) = G(z, m)G(z^{-1}, m)\phi_{xx}(z), \tag{3}$$

where $G(z, m)$ is the modified z transform associated with $G(s)$.

6) Assume that two linear systems $G_1(s)$ and $G_2(s)$ are subjected to input sequences $x_1^*(t)$ and $x_2^*(t)$, respectively, and that the system outputs in response to these inputs are $y_1(t)$ and $y_2(t)$ respectively. Then the "modified" cross pulse-spectral density for the output signals $y_1(t)$ and $y_2(t)$ is given by

$$\phi_{y_1y_2}(z, m) = G_1(z^{-1}, m)G_2(z, m)\phi_{x_1x_2}(z), \tag{4}$$

where $G_1(z^{-1}, m)$ and $G_2(z, m)$ are the modified z transforms associated with $G_1(-s)$ and $G_2(s)$, respectively, and $\phi_{x_1x_2}(z)$ is cross pulse-spectral density for input sequences $x_1^*(t)$ and $x_2^*(t)$.

7) If the response of linear system $G_1(s)$ to an input sequence $x_1^*(t)$ is $y_1(t)$, and the response of linear system $G_2(s)$ to an input signal $x_2(t)$ is $y_2(t)$, then the "modified" cross pulse-spectral density for the output signals $y_1(t)$ and $y_2(t)$ is given by

$$\phi_{y_1y_2}(z, m) = \hat{G}_1G_1(z, m)\phi_{x_1x_2}(z), \tag{5}$$

where $\hat{G}_1G_2(z, m)$ is the modified z transform associated with $G_1(-s)G_2(s)$.

The above properties of correlation sequences and pulse-spectral densities will find much use in the optimum design of digital control systems based upon the Wiener-Kolmogoroff theory.

Optimum Pulsed-Data Compensator for Minimum Mean-Square Sampled Error

The statistical design problem can be readily visualized by referring to Fig. 1. $G_0(z)$ is the over-all pulse-transfer function of a digital feedback control system which is to be optimized on the basis of minimum mean-square sampled error. The sampled input and output of the system are assumed to be

$$r(nT) = r_s(nT) + r_n(nT) \tag{6}$$

[6] H. S. Tsien, "Engineering Cybernetics," McGraw-Hill Book Co., Inc., New York, N. Y.; 1954.

[7] J. H. Laning and R. H. Battin, "Random Processes in Automatic Control," McGraw-Hill Book Co., Inc., New York, N. Y.; 1956.

[8] G. C. Newton, L. A. Gould, and J. F. Kaiser, "Analytical Design of Linear Feedback Control," John Wiley and Sons, Inc., New York, N. Y.; 1957.

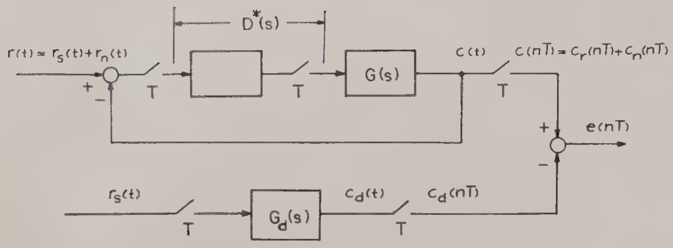

Fig. 1—Digital control system with sampled output.

and

$$c(nT) = c_s(nT) + c_n(nT), \tag{7}$$

respectively, where the subscript s refers to the signal and the subscript n refers to the random noise. Also assume that the desired pulse-transfer function be $G_d(z)$ and the desired output sequence be $c_d(nT)$. The error sequence of the system is then given by

$$\begin{aligned} e(nT) &= c(nT) - c_d(nT) \\ &= c_s(nT) + c_n(nT) - c_d(nT) \end{aligned} \tag{8}$$

and the mean-square sampled error is

$$\begin{aligned} \overline{e^2(nT)} = \overline{c_s{}^2(nT)} + \overline{c_n{}^2(nT)} + \overline{c_d{}^2(nT)} + \overline{c_s(nT)c_n(nT)} \\ + \overline{c_n(nT)c_s(nT)} - \overline{c_n(nT)c_d(nT)} - \overline{c_d(nT)c_n(nT)} \\ - \overline{c_s(nT)c_d(nT)} = \overline{c_d(nT)c_s(nT)}. \end{aligned} \tag{9}$$

In view of the relationships

$$\overline{x^2(nT)} = \phi_{xx}(0) = \frac{1}{2\pi j}\oint_\Gamma \phi_{xx}(z)z^{-1}dz \tag{10}$$

and

$$\overline{x(nT)y(nT)} = \phi_{xy}(0) = \frac{1}{2\pi j}\oint_\Gamma \phi_{xy}(z)z^{-1}dz, \tag{11}$$

where the contour of integration is the unit circle, the mean-square sampled error given in (9) may be written as

$$\begin{aligned} \overline{e^2(nT)} = \frac{1}{2\pi j}\oint_\Gamma [\phi_{c_sc_s}(z) + \phi_{c_nc_n}(z) + \phi_{c_dc_d}(z) + \phi_{c_sc_n}(z) \\ + \phi_{c_nc_s}(z) - \phi_{c_nc_d}(z) - \phi_{c_dc_n}(z) - \phi_{c_sc_d}(z) \\ - \phi_{c_dc_s}(z)]z^{-1}dz. \end{aligned} \tag{12}$$

By making use of the properties of the correlation sequences and the pulse-spectral densities given in (1) and (2), the above equation for the mean-square sampled error may be reduced to

$$\overline{e^2(nT)} = \frac{1}{2\pi j}\oint_\Gamma \phi_{ee}(z)z^{-1}dz, \tag{13}$$

in which

$$\begin{aligned} \phi_{ee}(z) = [G_0(z) - G_d(z)][G_0(z^{-1}) - G_d(z^{-1})]\phi_{r_sr_s}(z) \\ + G_0(z)[G_0(z^{-1}) - G_d(z^{-1})]\phi_{r_sr_n}(z) \\ + G_0(z^{-1})[G_0(z) - G_d(z)]\phi_{r_nr_s}(z) \\ + G_0(z)G_0(z^{-1})\phi_{r_nr_n}(z). \end{aligned} \tag{14}$$

With reference to Fig. 1, it is seen that the over-all pulse-transfer function is related to the pulse-transfer function of the controlled system $G(s)$ by

$$G_0(z) = W(z)G(z) \tag{15}$$

where

$$W(z) = \frac{D(z)}{1 - D(z)G(z)}, \tag{16}$$

and $D(z)$ is the pulse-transfer function of the pulsed-data compensator. Substituting (15) into (14) yields

$$\begin{aligned} \phi_{ee}(z) = [W(z)G(z) - G_d(z)][W(z^{-1})G(z^{-1}) - G_d(z^{-1})]\phi_{r_sr_s}(z) \\ + W(z)G(z)[W(z^{-1})G(z^{-1}) - G_d(z^{-1})]\phi_{r_sr_n}(z) \\ + W(z^{-1})G(z^{-1})[W(z)G(z) - G_d(z)]\phi_{r_nr_s}(z) \\ + W(z)W(z^{-1})G(z)G(z^{-1})\phi_{r_nr_n}(z). \end{aligned} \tag{17}$$

The design objective is the determination of the pulse-transfer functions $W(z)$ and $D(z)$ in such a way that the mean-square sampled error given by (13) is reduced to a minimum. It follows from (16) that $D(z)$ is given by

$$D(z) = \frac{W(z)}{1 - W(z)G(z)}. \tag{18}$$

The minimization problem can readily be solved by applying the calculus of variations to the integral of (13) with $\phi_{ee}(z)$ given in (17). To determine the condition for minimizing the mean-square sampled error, the pulse-transfer function $W(z)$ is given a small variation $\lambda\eta(z)$, and $W(z^{-1})$ is given a small variation $\lambda\eta(z^{-1})$. As a result the mean-square sampled error $\overline{e^2(nT)}$ undergoes a variation $\delta\overline{e^2(nT)}$. $\eta(z)$ is an arbitrary function of z, and λ is a parameter. Replacing $W(z)$ by its neighboring function $W(z)+\lambda\eta(z)$, and $W(z^{-1})$ by its neighboring function $W(z^{-1})+\lambda\eta(z^{-1})$, the mean-square sampled error becomes $\overline{e^2(nT)}+\delta\overline{e^2(nT)}$. Then $\overline{e^2(nT)}$ is a minimum, if

$$\left.\frac{d[\overline{e^2(nT)} + \delta\overline{e^2(nT)}]}{d\lambda}\right|_{\lambda=0} = 0. \tag{19}$$

Following the operations involved in (19) and simplifying leads to

$$\begin{aligned} \frac{1}{2\pi j}\oint_\Gamma \eta(z^{-1})G(z^{-1})\{W(z)G(z)\phi_{rr}(z) \\ - G_d(z)[\phi_{r_sr_s}(z) + \phi_{r_nr_s}(z)]\}z^{-1}dz \\ + \frac{1}{2\pi j}\oint_\Gamma \eta(z)G(z)\{W(z^{-1})G(z^{-1})\phi_{rr}(z) \\ - G_d(z^{-1})[\phi_{r_sr_s}(z) + \phi_{r_sr_n}(z)]\}z^{-1}dz = 0 \end{aligned} \tag{20}$$

where

$$\phi_{rr}(z) = \phi_{r_sr_s}(z) + \phi_{r_nr_n}(z) + \phi_{r_sr_n}(z) + \phi_{r_nr_s}(z). \tag{21}$$

To determine the optimum $W(z)$, the z transform $\phi_{rr}(z)$ is factored into two parts, one part having zeros and poles inside the unit circle and the other part having zeros and poles outside the unit circle. Thus

$$\phi_{rr}(z) = \phi_{rr}^{+}(z)\phi_{rr}^{-}(z). \tag{22}$$

The z transform $G(z)G(z^{-1})$ is similarly factored:

$$G(z)G(z^{-1}) = [G(z)G(z^{-1})]^{+}[G(z)G(z^{-1})]^{-} \tag{23}$$

where the first factor contains zeros and poles inside the unit circle and the second factor contains zeros and poles outside the unit circle. Then (20) may be written as

$$\frac{1}{2\pi j}\oint_{\Gamma} \eta(z^{-1})\phi_{rr}^{-}(z)[G(z)G(z^{-1})]^{-}\left\{[G(z)G(z^{-1})]^{+}W(z)\phi_{rr}^{+}(z) - \frac{G(z^{-1})G_d(z)[\phi_{r_sr_s}(z)+\phi_{r_nr_s}]}{[G(z)G(z^{-1})]^{-}\phi_{rr}^{-}(z)}\right\}z^{-1}dz$$
$$+\frac{1}{2\pi j}\oint_{\Gamma} \eta(z)\phi_{rr}^{+}(z)[G(z)G(z^{-1})]^{+}\left\{[G(z)G(z^{-1})]^{-}W(z^{-1})\phi_{rr}^{-}(z) - \frac{G(z)G_d(z^{-1})[\phi_{r_sr_s}(z)+\phi_{r_sr_n}(z)]}{[G(z)G(z^{-1})]^{+}\phi_{rr}^{+}(z)}\right\}z^{-1}dz = 0. \tag{24}$$

For physical realizability, $W(z)$ and $\eta(z)$ can have no pole outside the unit circle of the z plane, and $W(z^{-1})$ and $\eta(z^{-1})$ can have no pole inside the unit circle. Thus, the function

$$\eta(z^{-1})\phi_{rr}^{-}(z)[G(z)G(z^{-1})]^{-}$$

contains poles outside the unit circle only, and the function

$$\eta(z)\phi_{rr}^{+}(z)[G(z)G(z^{-1})]^{+}$$

contains poles inside the unit circle only. Consequently, the above equation may be further reduced to

$$\frac{1}{2\pi j}\oint_{\Gamma} \eta(z^{-1})\phi_{rr}^{-}(z)[G(z)G(z^{-1})]^{-}\left\{[G(z)G(z^{-1})]^{+}W(z)\phi_{rr}^{+}(z) - \left\{\frac{G(z^{-1})G_d(z)[\phi_{r_sr_s}(z)+\phi_{r_nr_s}(z)]}{[G(z)G(z^{-1})]^{-}\phi_{rr}^{-}(z)}\right\}_{+}\right\}z^{-1}dz$$
$$+\frac{1}{2\pi j}\oint_{\Gamma} \eta(z)\phi_{rr}^{+}(z)[G(z)G(z^{-1})]^{+}\left\{[G(z)G(z^{-1})]^{-}Wz^{-1})\phi_{rr}^{-}(z) - \left\{\frac{G(z)G_d(z^{-1})[\phi_{r_sr_s}(z)+\phi_{r_sr_n}(z)]}{[G(z)G(z^{-1})]^{+}\phi_{rr}^{+}(z)}\right\}_{-}\right\}z^{-1}dz = 0 \tag{25}$$

in which the symbol $\{\ \ \}_{+}$ designates the operation of picking the part of a function of z with poles inside the unit circle, and the symbol $\{\ \ \}_{-}$ designates the operation of picking the part with poles outside the unit circle. For instance, in $F(z)=\{F(z)\}_{+}+\{F(z)\}_{-}$, the terms with poles inside the unit circle are lumped in the (+) part, and the terms with poles outside the unit circle are collected in the (−) part.

Now, if $W(z)$ is the optimum pulse-transfer function for minimum mean-square sampled error, (25) must be satisfied for arbitrary $\eta(z)$. Thus the optimum $W(z)$ is given by

$$W(z) = \frac{\left\{\dfrac{G(z^{-1})G_d(z)[\phi_{r_sr_s}(z)+\phi_{r_nr_s}(z)]}{[G(z)G(z^{-1})]^{-}\phi_{rr}^{-}(z)}\right\}_{+}}{[G(z)G(z^{-1})]^{+}\phi_{rr}^{+}(z)}. \tag{26}$$

Once the pulse-transfer function $W(z)$ is determined, the required pulse-transfer function of the compensator $D(z)$ follows immediately from (18). If the pulse-transfer function $G(z)$ of the controlled system contains no zero and no pole outside the unit circle of the z plane,

$$[G(z)G(z^{-1})]^{+} = G(z) \tag{27}$$

and

$$[G(z)G(z^{-1})]^{-} = G(z^{-1}). \tag{28}$$

Consequently, under such conditions, (26) can be simplified to

$$W(z) = \frac{\left\{\dfrac{G_d(z)[\phi_{r_sr_s}(z)+\phi_{r_nr_s}(z)]}{\phi_{rr}^{-}(z)}\right\}_{+}}{G(z)\phi_{rr}^{+}(z)}. \tag{29}$$

Eqs. (26) and (29) form the working formulas, from which the optimum $W(z)$ can be readily determined. The determination of $W(z)$ and $D(z)$ simply involves the evaluation of z transforms and simple algebraic manipulations. Clearly, with the aid of these two equations, the design of digital control systems for minimum mean-square sampled error can be carried out systematically in a simple manner.

Optimum Pulsed-Data Compensator for Minimum Mean-Square Error

Now, a slightly different problem is to be considered. This section will present a procedure for the design of the pulsed-data compensator to minimize the mean-square error of the system. To determine the optimum pulse-transfer function in a way as simple and systematic as the design procedure presented above, use is made of the modified z-transform technique.

With reference to Fig. 2, $G_0(z, m)$ is the over-all modified pulse-transfer function of the digital feedback control system which is to be optimized on the basis of minimum mean-square error. The input and the output of the control system are assumed to be $r(t)=r_s(t)+r_n(t)$ and $c(t)=c_s(t)+c_n(t)$, respectively. It is also assumed that the desired system transfer function is $G_d(s)$ and the desired output signal is $c_d(t)$. The system error is then given by

$$\begin{aligned} e(t) &= c(t) - c_d(t) \\ &= c_s(t) + c_n(t) - c_d(t). \end{aligned} \tag{30}$$

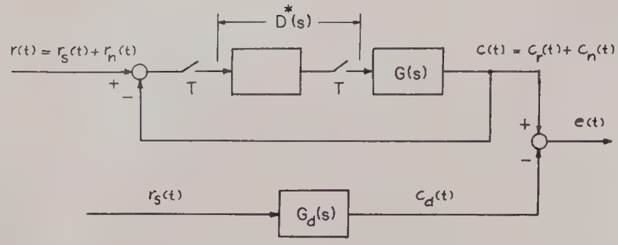

Fig. 2—Digital control system with continuous output.

In terms of samples,

$$e(nT, m) = c_s(nT, m) + c_n(nT, m) - c_d(nT, m). \quad (31)$$

It follows from (31) that the mean-square sampled-error for a particular value of m is given by

$$\begin{aligned}\overline{e^2(nT, m)} &= \overline{c_s{}^2(nT, m)} + \overline{c_n{}^2(nT, m)} + \overline{c_d{}^2(nT, m)} \\ &+ \overline{c_s(nT, m)c_n(nT, m)} + \overline{c_n(nT, m)c_s(nT, m)} \\ &- \overline{c_s(nT, m)c_d(nT, m)} - \overline{c_d(nT, m)c_s(nT, m)} \\ &- \overline{c_n(nT, m)c_d(nT, m)} - \overline{c_d(nT, m)c_n(nT, m)}. \quad (32)\end{aligned}$$

Following the argument of the previous section, (32) may be reduced to

$$\overline{e^2(nT, m)} = \frac{1}{2\pi j}\oint_\Gamma \phi_{ee}(z, m)z^{-1}dz, \quad (33)$$

in which

$$\begin{aligned}\phi_{ee}(z, m) &= \phi_{c_sc_s}(z, m) + \phi_{c_nc_n}(z, m) + \phi_{c_dc_d}(z, m) + \phi_{c_sc_n}(z, m) \\ &+ \phi_{c_nc_s}(z, m) - \phi_{c_sc_d}(z, m) - \phi_{c_dc_s}(z, m) \\ &- \phi_{c_nc_d}(z, m) - \phi_{c_dc_n}(z, m). \quad (34)\end{aligned}$$

In view of the properties of the pulse-spectral densities given in (3) and (5), $\phi_{ee}(z, m)$ may be expressed in terms of the modified z transforms of the system transfer functions. Thus,

$$\begin{aligned}\phi_{ee}(z, m) = &\Big[G_0(z, m)G_0(z^{-1}, m) - \hat{G}_0G_d(z, m) - G_0\hat{G}_d(z, m) \\ &+ \frac{G_r(z, m)}{\phi_{r_sr_s}(z)}\Big]\phi_{r_sr_s}(z) \\ &+ [G_0(z, m)G_0(z^{-1}, m) - G_0\hat{G}_d(z, m)]\phi_{r_sr_n}(z) \\ &+ [G_0(z, m)G_0(z^{-1}, m) - \hat{G}_0G_d(z, m)]\phi_{r_nr_s}(z) \\ &+ G_0(z, m)G_0(z^{-1}, m)\phi_{r_nr_n}(z) \quad (35)\end{aligned}$$

where $\hat{G}_0G_d(z, m)$, $G_0\hat{G}_d(z, m)$ and $G_r(z, m)$ are the modified z transforms associated with $G_0(-s)G_d(s)$, $G_0(s)G_d(-s)$ and $G_d(s)G_d(-s)\phi_{r_sr_s}(s)$, respectively.

Examination of Fig. 2 reveals that

$$G_0(s) = W^*(s)G(s) \quad (36)$$

where

$$W^*(s) = \frac{D^*(s)}{1 + D^*(s)G^*(s)}. \quad (37)$$

Taking the modified z transform of both sides of (36) yields

$$G_0(z, m) = W(z)G(z, m). \quad (38)$$

By substitution of (36) and (38), $\phi_{ee}(z, m)$ given in (35) becomes

$$\begin{aligned}\phi_{ee}(z, m) &= [W(z)W(z^{-1})G(z, m)G(z^{-1}, m) - W(z^{-1})\hat{G}G_d(z, m) \\ &- W(z)G\hat{G}_d(z, m) + G_r(z, m)/\phi_{r_sr_s}(z)]\phi_{r_sr_s}(z) \\ &+ [W(z)W(z^{-1})G(z,m)G(z^{-1},m) - W(z)G\hat{G}_d(z,m)]\phi_{r_sr_n}(z) \\ &+ [W(z)W(z^{-1})G(z,m)G(z^{-1},m) - W(z^{-1})\hat{G}G_d(z,m)]\phi_{r_nr_s}(z) \\ &+ W(z)W(z^{-1})G(z, m)G(z^{-1}, m)\phi_{r_nr_n}(z). \quad (39)\end{aligned}$$

It has been shown that the mean-square error $\overline{e^2(t)}$ can be determined from the mean-square error sequence $\overline{e^2(nT, m)}$ through integration;[5] that is,

$$\overline{e^2(t)} = \int_0^1 \overline{e^2(nT, m)}dm. \quad (40)$$

Combining (40) and (33) gives the mean-square error as

$$\overline{e^2(t)} = \frac{1}{2\pi j}\oint_\Gamma z^{-1}dz\int_0^1 \phi_{ee}(z, m)dm \quad (41)$$

where

$$\begin{aligned}\int_0^1 \phi_{ee}(z, m)dm &= [W(z)W(z^{-1})K_0(z) - W(z)K_2(z) - W(z^{-1})K_2(z^{-1}) \\ &+ K_3(z)/\phi_{r_sr_s}(z)]\phi_{r_sr_s}(z) \\ &+ [W(z)W(z^{-1})K_0(z) - W(z)K_2(z)]\phi_{r_sr_n}(z) \\ &+ [W(z)W(z^{-1})K_0(z) - W(z^{-1})K_2(z^{-1})]\phi_{r_nr_s}(z) \\ &+ W(z)W(z^{-1})K_0(z)\phi_{r_nr_n}(z) \quad (42)\end{aligned}$$

in which

$$K_0(z) = \int_0^1 G(z, m)G(z^{-1}, m)dm \quad (43)$$

$$K_2(z) = \int_0^1 G\hat{G}_d(z, m)dm \quad (44a)$$

$$K_2(z^{-1}) = \int_0^1 \hat{G}G_d(z, m)dm \quad (44b)$$

$$K_3(z) = \int_0^1 G_r(z, m)dm \quad (45)$$

Since (41) bears a close resemblance to (13) of the preceding section, the minimization problem may be solved in a similar fashion.

Application of the calculus of variations to the integral of (41) yields the optimum pulse-transfer function

$$W(z) = \frac{\left\{\dfrac{K_2(z^{-1})[\phi_{r_sr_s}(z) + \phi_{r_nr_s}(z)]}{K_0{}^-(z)\phi_{rr}{}^-(z)}\right\}_+}{K_0{}^+(z)\phi_{rr}{}^+(z)}. \quad (46)$$

Clearly, the pulse-transfer function of the required compensator $D(z)$ follows immediately from substituting (46) into (18). Eq. (46) can serve as the working formula for the statistical design of digital control systems based upon the minimum mean-square-error criterion. Since (46) involves simply the evaluation of z transforms and modified z transforms, ordinary integration and simple algebraic manipulations, the determination of the pulse-transfer function of the required compensator $D(z)$ is indeed a simple matter.

Illustrative Examples

Example 1: Design of digital compensator for minimizing mean-square error sequence.

Referring to Fig. 1, assume that

$$G(s) = \frac{1 - \epsilon^{-Ts}}{s^2(s+1)}$$

$$G_d(s) = 1 \qquad T = 0.2 \text{ second}$$

$$\phi_{r_s r_s}(\omega) = \frac{1}{1+\omega^2} \qquad \phi_{r_n r_n}(\omega) = 0.1$$

$$\phi_{r_s r_n}(\omega) = \phi_{r_n r_s}(\omega) = 0.$$

Determine the transfer function of the digital compensator required for minimizing the mean-square error sequence.

The design is initiated with the determination of the z transform associated with $G(s)$, $\phi_{r_s r_s}(s)$, $\phi_{r_n r_n}(s)$, and $\phi_{rr}(s)$. It is found that

$$G(z) = \frac{0.019(z+0.895)}{(z-1)(z-0.819)}$$

$$\phi_{r_s r_s}(z) = \frac{-0.2013z}{(z-0.819)(z-1.221)}$$

$$\phi_{r_n r_n}(z) = 0.1$$

$$\phi_{rr}(z) = \frac{0.1(z-0.261)(z-3.791)}{(z-0.819)(z-1.221)}.$$

From the above equations, one obtains

$$\phi_{rr}^+(z) = \frac{0.316(z-0.261)}{(z-0.819)}$$

$$\phi_{rr}^-(z) = \frac{0.316(z-3.791)}{(z-1.221)}$$

and

$$\left\{\frac{\phi_{r_s r_s}(z)}{\phi_{rr}^-(z)}\right\}_+ = \frac{0.176}{z-0.819}.$$

Making use of (26),

$$W(z) = \frac{29.25(z-1)(z-0.819)}{(z-0.261)(z+0.895)}.$$

Substituting into (18) yields the transfer function of the required compensator as

$$D(z) = \frac{29.25(z^2 - 1.819z + 0.819)}{(z^2 - 0.08z - 0.73)}.$$

Example 2: Design of digital compensator for minimizing the mean-square error.

Consider the system shown in Fig. 2. Assume that

$$G(s) = \frac{1-\epsilon^{-Ts}}{s(s+1)}$$

$$G_d(s) = 1 \qquad T = 0.2 \text{ second}$$

$$\phi_{r_s r_s}(\omega) = \frac{1}{1+\omega^2} \qquad \phi_{r_n r_n}(\omega) = 0.1$$

$$\phi_{r_s r_n}(\omega) = \phi_{r_n r_s}(\omega) = 0.$$

Determine the transfer function of the digital compensator required for minimizing the mean-square error.

To design the system by following the above procedures, the z transforms $K_0(z)$ and $K_2(z^{-1})$ are first to be determined. Since the modified z transform associated with $G(s)$ is

$$G(z, m) = \frac{1}{z} - \frac{(z-1)\epsilon^{-0.2m}}{z(z-0.819)}$$

$$K_0(z) = \int_0^1 G(z, m)G(z^{-1}, m)dm$$

$$= \frac{-0.0069(z-0.305)(z+3.274)}{(z-0.819)(z-1.22)}$$

and

$$K_2(z^{-1}) = \int_0^1 G(z^{-1}, m)dm$$

$$= \frac{-0.107z(z+1.07)}{(z-1.22)}.$$

Factoring $K_0(z)$ yields

$$K_0^+(z) = \frac{0.083(z+0.305)}{(z-0.819)}$$

$$K_0^-(z) = \frac{-0.083(z+3.274)}{(z-1.22)}.$$

Then

$$\left\{\frac{K_2(z^{-1})\phi_{r_s r_s}(z)}{K_0^-(z)\phi_{rr}^-(z)}\right\}_+ = \frac{0.0871}{z-0.819}.$$

It follows from (46) that

$$W(z) = \frac{3.32(z-0.819)}{(z-0.261)(z+0.305)}.$$

Therefore, the transfer function of the required compensator is given by

$$D(z) = \frac{3.32(z - 0.819)}{(z^2 + 0.044z - 0.669)}.$$

Conclusions

This paper introduces a simplified procedure for statistical design of digital control systems. The design techniques presented above are based upon two performance criteria, the minimization of mean-square sampled error and the minimization of mean-square error. These two criteria apply to purely digital control systems and digital-controlled continuous-data systems, respectively. The input signals and noise are assumed to be stationary random functions. In this paper the optimization problem is treated systematically by use of the modified z-transform technique. To simplify the design procedure, use is made of some important properties of correlation sequence and pulse-spectral density, which are summarized above. Based upon the above two performance criteria, general design formulas are derived, which can readily be applied to the determination of the pulse-transfer functions of optimum digital compensators. In deriving these design formulas, special attention is paid to physical realizability, nonminimum-phase type transfer functions and system constraints. Following the procedures presented above, the statistical design of digital control systems is made no more difficult than the statistical design of conventional continuous-data control systems. Discussions reported in this paper are concerned only with digital control systems having noise occurring in the input channel. The optimum design of digital control systems with noise or disturbance occurring at other points of the control loop will be discussed in a forthcoming paper.

Discussion

D. P. Lindorff, University of Connecticut, Storrs, Conn. The author has presented a clear and detailed derivation of the optimum pulsed filter in the complex-frequency domain, imposing various constraints. It may prove of some interest to point out that the results obtained can be derived directly by using the Bode-Shannon approach.[1]

Thus in the model shown in Fig. 3, which is equivalent to that of Fig. 1, the optimum filter without regard to realizability is seen to be

$$G_{0_{\text{opt}}}(z) = \frac{G_d(z)\phi_{r_s r_s}(z)}{\phi_{rr}(z)}. \tag{47}$$

In this discussion $\phi_{r_s r_n}(z) = 0$ for simplicity.

[1] H. W. Bode and C. E. Shannon, "A simplified derivation of linear least-square smoothing and prediction theory," Proc. IRE, vol. 38, pp. 423–424; April, 1950.

The optimum realizable filter is then determined directly to be

$$G_{0_{\text{opt}}}(z) = \left[\frac{1}{\phi_{rr}^{+}(z)}\right]\left[\frac{G_d(z)\phi_{r_s r_s}(z)}{\phi_{rr}^{-}(z)}\right]_{+}, \tag{48}$$

where the term represented by the first factor is chosen to produce white noise. The term represented by the second factor is then taken to be the realizable component of the filter, which would produce the over-all optimum filter without regard to realizability.

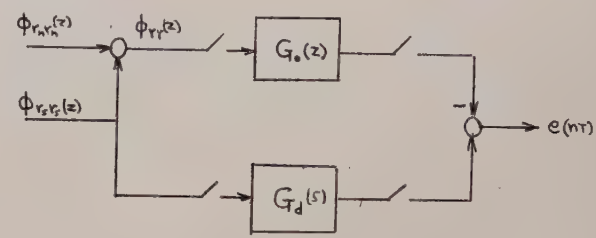

Fig. 3.

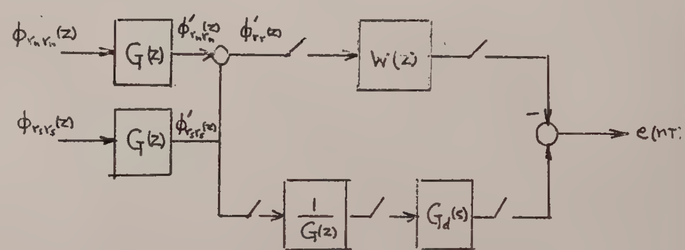

Fig. 4.

If constraints are imposed upon $G(z)$, resulting in (26), wherein $G_0(z) = W(z)G(z)$, the desired expression for $W(z)$ can be obtained by the Bode-Shannon approach, if the diagram of Fig. 3 is redrawn in the form shown in Fig. 4. Considering $\phi_{r_s r_s}'(z)$ and $\phi_{r_n r_n}'(z)$ as inputs, the realizable optimum expression for $W(z)$ is seen to be

$$W_{\text{opt}}(z) = \left[\frac{1}{\phi_{rr}'^{+}(z)}\right]\left[\frac{G_d(z)}{G(z)}\,\frac{\phi_{r_s r_s}'(z)}{\phi_{rr}'^{-}(z)}\right]_{+}, \tag{49}$$

where it is understood that $G_d(z)/G(z)$ replaces $G_d(z)$ in Fig. 3. By using the following relationships, which are evident from Fig. 4,

$$\phi_{r_s r_s}'(z) = \phi_{r_s r_s}(z)G(z)G(z^{-1})$$
$$\phi_{r_n r_n}'(z) = \phi_{r_n r_n}(z)G(z)G(z^{-1})$$
$$\phi_{rr}'(z) = \phi_{rr}(z)G(z)G(z^{-1}),$$

(49) becomes

$$W_{\text{opt}}(z) = \left[\frac{1}{\phi_{rr}^{+}(z)[G(z)G(z^{-1})]^{+}}\right]\left[\frac{G_d(z)G(z^{-1})\phi_{r_s r_s}(z)}{\phi_{rr}^{-}(z)[G(z)G(z^{-1})]^{-}}\right]_{+} \tag{50}$$

which agrees with (26).

The important case represented by Fig. 2, in which the G_d channel is continuous, is also amenable to the Bode-Shannon approach if a mathematical artifice which will be termed a *reciprocal sampler* is introduced. With this device it is permissible to rearrange Fig. 2 into the form of Fig. 5.

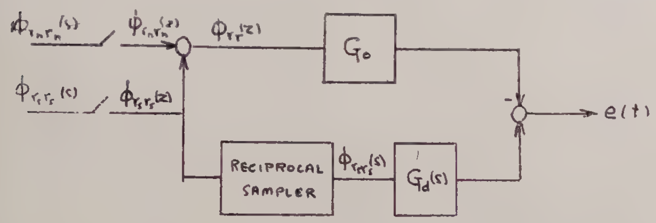

Fig. 5.

There should be no concern as to the realizability of the reciprocal sampler, since the G_d channel has no realizability constraints imposed upon it. Apparently the transfer function of the reciprocal sampler is equal to $\phi_{r_s r_s}(s)/\phi_{r_s r_s}(z)$. The optimum realizable filter can now be formulated by inspection, treating $\phi_{r_s r_s}(z)$ and $\phi_{r_n r_n}(z)$ as inputs, and considering the equivalent desired filter to be given by $G_d' = G_d \phi_{r_s r_s}(s)/\phi_{r_s r_s}(z)$. The optimum filter without regard to realizability is then

$$G_{0_{\text{opt}}}(s) = \frac{\phi_{r_s r_s}(z)}{\phi_{rr}(z)} G_d'(s, z). \tag{51}$$

Imposing realizability constraints on (51) according to the Bode Shannon approach, it is seen that

$$G_{0_{\text{opt}}}(s) = \left[\frac{1}{\phi_{rr}^{+}(z)}\right]\left[\frac{\phi_{r_s r_s}(s) G_d(s)}{\phi_{rr}^{-}(z)}\right]_{+}. \tag{52}$$

It will be perceived that the simple manipulation used in deriving Fig. 4 cannot be applied to Fig. 5, with the consequence that (46) does not appear to follow directly from (52).

The purpose of this discussion has been to investigate the applicability of the Bode-Shannon technique to pulsed filters, thereby supplementing the mathematical derivations with the conceptual appeal offered by this approach.

Author's Comment: The author appreciates the discussion by Prof. D. P. Lindorff. Because of the resemblance between transfer functions and pulse-transfer functions, correlation functions and correlation sequences, spectral densities and pulse-spectral densities, the extension of the Bode-Shannon approach to the design of digital control systems for minimum mean-square error sequence is fairly straightforward. Following the method of Bode and Shannon, the author has also checked the result given by (26). In fact, if one is versed in the statistical design theory for continuous-data control systems, (26) may be written down by inspection of the system block digram. The derivation of (26) is presented simply to illustrate the design procedure for digital control systems based upon the Wiener-Kolmogoroff theory.

Difficulties are encountered, however, when the Bode-Shannon method is applied to the more important problem of designing digital control systems for minimum mean-square error. This point has also been noted by Prof. Lindorff in his discussion. The operation involved in (52) of Prof. Lindorff's discussion seems rather difficult to perform, since the term in the operating brackets is a function of both s and z. In view of the difficulties which arise in treating more important cases, the Bode-Shannon approach is not discussed in this paper. On the other hand, the design approach presented in this paper is quite general, and can be applied to other system configurations than that discussed above. This technique has been successively applied to digital control systems, with noise or disturbance occurring at other points of the control loop than in the input channel of the system. The main purpose of this paper is to introduce a simplified approach for the statistical design of digital control systems of various configurations.

Determination of Periodic Modes in Relay Servomechanisms Employing Sampled Data*

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Summary—**A difference equation approach is presented to determine the periodic modes in relay servomechanisms employing sampled data. In using the method, together with Izawa and Weaver's method, the investigator can have all the information concerning possible periodic modes of a system.**

This new technique is based on expressing periodic sequences in terms of orthogonal functions in the discrete sense and balancing the harmonics.

SAMPLED-DATA feedback systems containing a relay and a zero-order hold have been extensively investigated. The describing function approach for these systems was used by Chow[1] and Russell[2] to approximate the continuous output for sinusoidal inputs. Altar and Helstrom,[3] Kalman,[4,5] Izawa and Weaver,[6] Izawa,[7,8] and Mullin and Jury[9] employed the phase-plane approach to study such systems.

The describing-function approach serves well to approximate the possible oscillations due to sinusoidal inputs, but it remains to be seen whether or not the findings are exhaustive. The phase-plane approach can reveal all possible periodic modes. However, this can only be achieved through step-by-step computations and is generally only applicable to second-order systems.

Izawa and Weaver[6] developed a simple approach to determine all possible periods of cyclic response of a relay sampled-data feedback system with given relay characteristics and sampling period.

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[1] C. K. Chow, "Contactor Servomechanism Employing Sampled-Data," Ph.D. dissertation, Cornell University, Ithaca, N. Y.; 1953.

[2] C. K. Chow, "Contactor servomechanism employing sampled-data," *Trans. AIEE*, vol. 73, pt. 2, pp. 51–64; March, 1954. Also see F. A. Russell, "Discussion to Chow's paper," *Trans. AIEE*, vol. 73, pt. 2, pp. 62–64; March, 1954.

[3] W. Altar and C. W. Helstrom, "Phase-plane Representation of Sampled-Servomechanisms," Westinghouse Electric Corp., Pittsburgh, Pa., Res. Rept. No. 60-94410-14-B; September, 1953.

[4] R. E. Kalman, "Phase-Plane Analysis of Non-Linear Sampled-Data Servomechanisms," M.S. Thesis, Mass. Inst. Tech., Cambridge, Mass.; June, 1954.

[5] R. E. Kalman, "Some aspects of non-linear sampled-data systems," *Proc. Symp. on Non-Linear Circuit Analysis*, Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.; April, 1956.

[6] K. Izawa and L. A. Weaver, "Relay-type feedback control systems, with dead time and sampling," *Trans. AIEE*, vol. 78, pt. 2; May, 1959.

[7] K. Izawa, "On-off control with periodic sensing device," *Trans. ASME*, vol. 80, pp. 1459–1464; November, 1959.

[8] K. Izawa, discussion following reference 9.

[9] F. J. Mullin and E. I. Jury, "A phase-plane approach to relay sampled-data feedback systems," *Trans. AIEE* (*Applications and Industry*), no. 40, pp. 517–523; January, 1959.

In this paper the difference equation approach is used, together with Izawa and Weaver's findings, to reveal all possible periodic modes of relay sampled-data feedback systems of any order in a simplified way.

The effect of the presence of a dead zone in the relay characteristic is studied. The steady-state response of the system corresponding to step, ramp, or certain sinusoidal inputs is also discussed.

Difference Equation Representation

The system to be considered is shown in Fig. 1. The linear portion of the system, with transfer function $G(s)$, has p poles. The relay is defined by the following relationship

$$\begin{aligned} f_n &= 1 & e_n &\geq e \\ &= 0 & |e_n| &\leq e \\ &= -1 & e_n &\leq -e. \end{aligned} \tag{1}$$

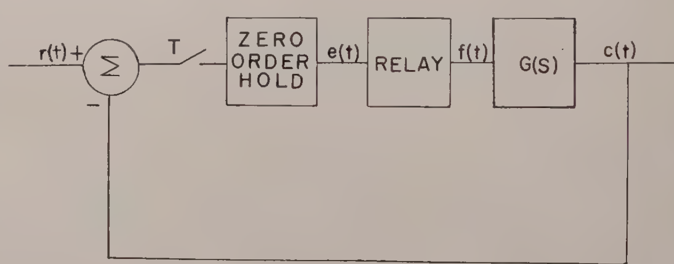

Fig. 1—A relay sampled-data control system.

The difference equation relating c_n and f_n, the system output and the relay output of the system at the nth sampling instant respectively, can be written as

$$a_0c_{n+p} + a_1c_{n+p-1} + a_2c_{n+p-2} + \cdots + a_pc_n = b_0f_{n+p} + b_1f_{n+p-1} + b_2f_{n+p-2} + \cdots + b_pf_n. \tag{2}$$

The derivation of (2) can be accomplished either by the direct integration or from z-transform tables. The order on the right-hand side is at most equal to that of the left-hand side. This is due to the presence of a zero-order hold in front of the relay.

Physical Considerations

When a periodic mode of period NT, where T is the sampling period, exists in a relay sampled-data control system, the output of the system at sampling instants will repeat N values successively; N is an integer.

As the periodic output is fed back and sampled, the output of the zero-order hold at the sampling instants will also repeat a set of N values. Such a set of numbers is defined as a sequence. This sequence, when passed through the relay, is quantized into a periodic one which consists of terms whose magnitudes can only be $+1$, 0, and -1. That is to say, the output of the relay at sampling instants can be represented as a periodic sequence, composed of values 1, 0, and -1.

Expansion of Periodic Sequences in Terms of Orthogonal Functions in the Discrete Sense

In investigating the oscillations that might occur in a conventional feedback control system, the possible periodic output of the system can be expressed as a Fourier series with an infinite number of terms. This often makes the complete evaluation impossible.

In the present study a repetitive discrete sequence, representing the output of the system at sampling instants, is to be determined. One question which will be raised is whether or not an expression with a finite number of terms can be found to represent the discrete sequence.

It has been established that this can be done in terms of orthogonal functions in the discrete sense. A set of functions $S_1(i) \cdots S_n(i)$ with integral argument, i, defined when

$$n_1 \leq i \leq n_2$$

is said to be orthogonal in the discrete sense, if

$$\sum_{i=n_1}^{n_2} S_l(i) S_m(i) = 0 \qquad l \neq m. \tag{3}$$

A periodic sequence $P(i)$ of period N can be written in one and only one way as a linear function of $S_1, S_2, \cdots, S_N$ with constant coefficients.[10]

Trigonometric functions with integral argument form an orthogonal set. In terms of this set of orthogonal functions, the periodic sequence can be expressed as follows:

$$P(i) = \sum_{n=0}^{k} \left(a_n \cos n \frac{2\pi i}{N} + b_n \sin n \frac{2\pi i}{N} \right)$$

if $N = 2k + 1$,

$$P(i) = \sum_{n=0}^{k} \left(a_n \cos n \frac{2\pi i}{N} + b_n \sin n \frac{2\pi i}{N} \right) + a_{k+1} \cos \pi i$$

if $N = 2k + 2$, (4)

[10] T. Fort, "Finite Differences and Difference Equations in the Real Domain," Oxford University Press, New York, N. Y.; 1948.

where

$$a_n = \frac{2}{N} \sum_{m=0}^{N-1} P(m) \cos n \frac{2\pi m}{N}, \qquad a_0 = \frac{1}{N} \sum_{m=0}^{N-1} P(m),$$

$$b_n = \frac{2}{N} \sum_{m=0}^{N-1} P(m) \sin n \frac{2\pi m}{N}, \qquad a_{k+1} = \frac{1}{N} \sum_{m=0}^{N-1} P(m) \cos \pi m,$$

$$n = 1, \cdots k. \tag{5}$$

Determination Process

If an oscillation of period NT prevails in a relay sampled-data control system, the output of the system will repeat a set of N values. Its output at the nth sampling instant can be written as

$$c_n = \sum_{r=0}^{k} \left[v_r \cos r \frac{2\pi n}{N} + q_r \sin r \frac{2\pi n}{N} \right] + v_{k+1} \cos \pi n, \tag{6a}$$

where $N = 2k+2$; and

$$c_n = \sum_{r=0}^{k} \left[v_r \cos r \frac{2\pi n}{N} + q_r \sin r \frac{2\pi n}{N} \right], \tag{6b}$$

where $N = 2k+1$.

The output of the relay at the nth sampling instant, f_n, can be written as

$$f_n = \sum_{r=0}^{k} \left[g_r \cos r \frac{2\pi n}{N} + h_r \sin r \frac{2\pi n}{N} \right] + g_{k+1} \cos \pi n, \tag{7a}$$

where $N = 2k+2$; and

$$f_n = \sum_{r=0}^{k} \left[g_r \cos r \frac{2\pi n}{N} + h_r \sin r \frac{2\pi n}{N} \right], \tag{7b}$$

where $N = 2k+1$.

In determining the possible periodic modes, the value of N is first assigned. One question that will be raised here is that, since N is arbitrarily assigned, the task of identifying the periodic modes will be endless. The fact is that the possible values of N for a relay sampled-data system is limited and can be readily evaluated by Izawa and Weaver's approach.[6,8] They are determined by the half-width of the dead zone and the sampling period T. In an example to follow, the possible values of N are only 2, 4 and 6.[8]

Once the value of N is determined, the periodic output sequence of the relay can be postulated. It seems that even with the value of N specified, there are numerous possibilities for such a postulation. It is actually not the case. A systematic approach is presented in the Appendix to determine possible sequences that have to be considered for a specified N.

The postulated sequence, by means of (5), will determine the coefficients $g_0, g_1, \cdots, g_k, g_{k+1}, h_1, h_2, \cdots, h_k$ in (7a) and (7b).

In Appendix I, sets of values of $g_0, \cdots, g_{k+1}, h_1, \cdots, h_k$ are calculated corresponding to possible sequences. They are ready for universal use as long as the investigator normalizes the output of the relay to $+1$ and -1.

Substituting (6a) and (7a) or (6b) and (7b) into (2), and collecting terms, one obtains

$$\sum_{l=0}^{p} a_{p-l}\left[v_r \cos\frac{2\pi r}{N} l + q_r \sin\frac{2\pi r}{N} l\right] = \sum_{l=0}^{p} b_{p-l}\left[g_r \cos\frac{2\pi r}{N} l + h_r \sin\frac{2\pi r}{N} l\right] \quad r = 0, 1, \cdots, k+1, \tag{8}$$

$$\sum_{l=0}^{p} a_{p-l}\left[q_r \cos\frac{2\pi r}{N} l - v_r \sin\frac{2\pi r}{N} l\right] = \sum_{l=0}^{p} b_{p-l}\left[h_r \cos\frac{2\pi r}{N} l - g_r \sin\frac{2\pi r}{N} l\right] \quad r = 1, 2, \cdots, k. \tag{9}$$

It should be pointed out here that these N equations are linear. Furthermore, for each r, v_r and q_r are related in *two* and *only two* equations. They can be easily solved. The determination of the N unknowns enables the periodic output sequence to be evaluated from (6). The postulated output of the system at the sampling instants will be $c_1, c_2, \cdots, c_N$. This is done by assigning values of n in (6) to be $1, 2, \cdots, N$ respectively.

The final step is to feed $c_1, c_2, \cdots, c_N$ back to the input. If the relay output sequence so produced is the same as that upon which the coefficients in (7) are determined, then there is certainly a periodic oscillation which is defined by $c_1, c_2, \cdots, c_N$.

If the relay output sequence so produced is *not* the same as that postulated, then it is decisively clear that there is no such oscillation possible in the servo system.

Illustrative Example

The system shown in Fig. 1 is to be studied, in which the sampling period $T=1$ second, $G(s)=1/s(s+1)$, and the relay is ideal. (This is the system studied by Chow,[1] Kalman,[4,5] and Mullin and Jury.[9]) Even though the problem here discussed is of second order, it should again be emphasized that the method is equally applicable to systems of any order.

The difference equation relating c_n and f_n can be found by direct integration or from a z-transform table to be

$$c_{n+2} - 1.368c_{n+1} + 0.368c_n = 0.368f_{n+1} + 0.264f_n.$$

That is

$$\begin{aligned} &a_0 = 1 \qquad a_1 = -1.368 \qquad a_2 = 0.368 \\ &b_0 = 0 \qquad b_1 = 0.368 \qquad b_2 = 0.264, \\ &p = 2. \end{aligned} \tag{10}$$

The first task is to determine possible values of N. In the system to be studied, N can only be 2, 4, and 6.[8]

1) Let $N=2$.

From (10), it can be seen that

$$\sum_{l=0}^{p} a_{p-l} = 0.$$

It should be remarked here that

$$\sum_{l=0}^{p} a_{p-l} = 0$$

is not a coincidence. It is due to the fact that the transfer function of the plant has a factor $1/s$.

Referring to Table II in the Appendix, the possible periodic output sequence of the relay can only be $1, -1$. The two coefficients g_0 and g_1 are 0 and 1 respectively.

The output of the relay at the nth sampling instant is

$$f_n = \cos \pi n. \tag{11}$$

As $N=2$, the output of the system at the nth sampling instant can be written

$$c_n = v_0 + v_1 \cos \pi n. \tag{12}$$

From (10), (11), (12), and (8), the following two equations are obtained:

$$\begin{aligned} r = 0 \quad & (0.368 - 1.368 + 1)v_0 = 0 \\ r = 1 \quad & (0.368 + 1.368 + 1)v_1 = 0.264 \times 1 - 0.368 \times 1 \\ & = -0.104. \end{aligned} \tag{13}$$

v_0 and v_1 are evaluated from (13). Eq. (12) can be written as

$$c_n = v_0 - 0.038 \cos \pi n. \tag{14}$$

Eq. (13) shows that v_0 can be chosen arbitrarily, but it is not actually the case; v_0 should be chosen in such a way that the output of the system, when fed back to the input, will produce the postulated relay output sequence $1, -1$.

If v_0 is so chosen that

$$-0.038 < v_0 < 0.038, \tag{15}$$

the condition to produce the posutlated periodic relay output sequence $1, -1$ will be met. This shows that the system can have the oscillation as defined by (14) and (15).

2) Let $N=4$.

From Table II in the Appendix, the only possible periodic sequence of the relay is $1, 1, -1$ and -1. The four coefficients g_0, g_1, g_2 and h_1 are 0, 1, 0 and 1 respectively.

The following two expressions can be obtained

$$f_n = \cos\frac{2\pi}{4}n + \sin\frac{2\pi}{4}n = \cos\frac{\pi}{2}n + \sin\frac{\pi}{2}n, \quad (16)$$

$$c_n = v_0 + v_1\cos\frac{\pi}{2}n + v_2\cos\pi n + q_1\sin\frac{\pi}{2}n. \quad (17)$$

From (10), (16), (17), (8) and (9), a set of four equations is obtained:

$$(1 - 1.368 + 0.368)v_0 = 0 \quad (18a)$$

$$-0.632v_1 - 1.368q_1 = 0.632 \quad (18b)$$

$$1.368v_1 - 0.632q_1 = -0.104 \quad (18c)$$

$$(1 + 1.368 + 0.368)v_2 = 0. \quad (18d)$$

From (18d)

$$v_2 = 0.$$

Solve (18b) and (18c) for v_1 and q_1 (v_1 and q_1 are related by two and only two equations,

$$v_1 = -0.238$$

$$q_1 = -0.352.$$

The output of the system at the nth sampling instant is

$$c_n = v_0 - 0.238\cos\frac{\pi}{2}n - 0.352\sin\frac{\pi}{2}n. \quad (19)$$

Considering that c_n of (19) should produce the postulated relay output sequence, it is clear that

$$-0.238 < v_0 < 0.238.$$

The results obtained above check with those obtained by Kalman.

Dead Zone

The presence of a dead zone in the relay characteristic introduces a new output level 0. The periodic mode can also be determined by the above procedure, as will be shown by the following example.

In the illustrative example, if the relay characteristic is defined as

$$\begin{aligned} f_n &= 1 & e_n &\geq 0.1 \\ &= 0 & |e_n| &< 0.1 \\ &= -1 & e_n &\leq -0.1. \end{aligned} \quad (20)$$

The possible values of N can be obtained from the Izawa and Weaver method.

Let $N=4$;

$$f_n = 0,\ 1,\ 0,\ -1 \text{ (postulated).}$$

Since $N=4$, there are 4 constants g_0, g_1, g_2 and h_1 to be determined. They could be determined by using (5) and the postulated f_n; but, by inspection, it is easily known that f_n can be written as

$$f_n = \sin\frac{\pi}{2}n. \quad (21)$$

The output of the system at the nth sampling instant can be written as

$$c_n = v_0 + v_1\cos\frac{\pi}{2}n + v_2\cos\pi n + q_1\sin\frac{\pi}{2}n. \quad (22)$$

From (10), (21), (22), (8) and (9), the following set of equations is obtained:

$$(1 - 1.368 + 0.368)v_0 = 0 \quad (23a)$$

$$-0.632v_1 - 1.368q_1 = 0.368 \quad (23b)$$

$$1.368v_1 - 0.632q_1 = 0.264 \quad (23c)$$

$$(1 + 1.368 + 0.368)v_2 = 0. \quad (23d)$$

From (23d),

$$v_2 = 0.$$

Solve (23b) and (23c)

$$v_1 = 0.057$$

$$q_1 = -0.295.$$

Eq. (22) can then be written as

$$c_n = v_0 + 0.057\cos\frac{\pi}{2}n - 0.295\sin\frac{\pi}{2}n.$$

The arbitrary constant v_0 can be determined by considering the relay characteristic, shown in (20), and the postulated relay output sequence. It is found that when

$$-0.043 < v_0 < 0.043,$$

the postulated relay output sequence can be reproduced.

Steady-State Response Due to Various Inputs

The examples above illustrate the procedure to determine the periodic modes of relay sampled-data systems due to self-excitation. This approach can also be applied to systems subjected to step, ramp, or certain sinusoidal inputs. This is accomplished by adjusting the constant "v_0" to ensure that the postulated periodic relay output will prevail.

Step Input

If a step input of magnitude m is applied, the possible periodic system output sequence can be expressed as

$$c_n = m + v_0 + \sum_{r=1}^{k}\left(v_r\cos\frac{2\pi n}{N} + q_r\sin\frac{2\pi n}{N}\right) + v_{k+1}\cos\pi n. \quad (24)$$

Suppose in the system studied in Examples 1 and 2, a step input of magnitude 1 is applied, then from (19) and (24), the possible steady-state response of the system will be

$$c_n = 1 + v_0 - 0.238 \cos \frac{\pi}{2} n - 0.352 \sin \frac{\pi}{2} n,$$

where $-0.238 < v_0 < 0.238$.

This result checks with that obtained by Mullin and Jury.

Ramp Input

If a ramp input of "mt" is applied where m is a constant, the possible periodic system output sequence can be expressed as

$$c_n = mnT + v_0 + \sum_{r=1}^{k} \left(v_r \cos \frac{2\pi n}{N} + q_r \sin \frac{2\pi n}{N} \right) + v_{k+1} \cos \pi n. \tag{25}$$

Certain Sinusoidal Inputs

When a sinusoidal input with a period $P = T/b$, where b is an integer, is applied, the system is essentially excited by a step input. If the following relationship exists, namely $Pb = T\omega$, then the system is excited by a repetitive sequence of period ω. The sequence can be expanded in the same fashion and the term "v_0" can be replaced by this expansion and v_0.

The procedure can only be carried out when there is an arbitrary constant available. That is to say, the transfer function of the linear plant has a factor $1/s$.

Conclusion

A unique difference equation approach is presented to determine the periodic modes in a relay sampled-data control system of any order. This new technique is based on expressing periodic sequences in terms of orthogonal functions in the discrete sense and balancing the harmonics. It enables the investigator to determine decisively whether a specific periodic mode exists or not, and if it does, to identify it exactly. In using this method together with Izawa and Weaver's method, the investigator can have all the information concerning possible periodic modes of a system.

This approach is extended to the case where the relay has a dead zone in its characteristic.

This approach is also extended to determine the possible steady-state response when the system is subjected to a step, a ramp or a certain sinusoidal input.

This approach is equally applicable to systems with different forms of feedback other than unity. If an element $H(s)$ is in the feedback path, it can be combined with $G(s)$ to form $G(s) = G(s)H(s)$. The same procedure can then be applied.

Appendix

The Derivation of Possible Periodic Output Sequences of an Ideal Relay

Let the number of output levels of a relay element be y and the period of the oscillation be NT; the number of possible periodic output sequences M of the relay will be y^N.

In the case of an ideal relay, $y=2$, $M=2^N$. When N increases, it seems that M will be prohibitively large. However, the cases that have to be investigated are quite limited, as shown in Table I.

TABLE I

N	Number of possible periodic sequences	Number of sequences that have to be investigated	Number of sequences to be considered when the plant transfer function has an integrating factor
2	4	1	1
4	16	2	1
6	64	5	2
8	256	16	5

The derivation of Table I and the approach to expand it can be presented by considering the case, $N=6$. M, the number of possible periodic sequences, $=2^6=64$.

These 64 cases can be classified into the following groups: Each sequence has

1) Three $(+1)s$ and three $(-1)s$

$$\frac{6.5.4}{3!} = 20 \text{ cases},$$

2) four $(+1)s$ and two $(-1)s$

$$\frac{6.5}{2!} = 15 \text{ cases},$$

3) two $(+1)s$ and four $(-1)s$

$$\frac{6.5}{2!} = 15 \text{ cases},$$

4) five $(+1)s$ and one $(-1)s$

$$\frac{6}{1} = 6 \text{ cases},$$

5) one $(+1)$ and five $(-1)s$

$$\frac{6}{1} = 6 \text{ cases},$$

6) six $(+1)s$

1 case,

7) six $(-1)s$

1 case.

The six cases in group 4) are

$$\begin{array}{rrrrrr} 1, & 1, & 1, & 1, & 1, & -1 \\ -1, & 1, & 1, & 1, & 1, & 1 \\ 1, & -1, & 1, & 1, & 1, & 1 \\ 1, & 1, & -1, & 1, & 1, & 1 \\ 1, & 1, & 1, & -1, & 1, & 1 \\ 1, & 1, & 1, & 1, & -1, & 1. \end{array}$$

By examining them carefully, one can see that, so far as periodic modes are concerned, they actually represent one sequence, say 1, 1, 1, 1, 1, −1. It is here defined that such a sequence is called a seed sequence. It is obvious that for an oscillation of period N, a seed sequence represents N possible sequences.

The same argument can be applied to group 5). Furthermore, group 5) can be obtained by replacing +1 by −1, −1 by 1 in group 4). It is defined here that group 5) is a reflection of group 4). Actually groups 4) and 5) present just *one* case to be investigated.

$$1, \quad 1, \quad 1, \quad 1, \quad 1, \quad -1. \tag{26}$$

Groups 6) and 7) represent actually an oscillation of period 1.

As group 2) is the reflection of group 3), only one has to be considered, say group 2). This group presents two seed sequences. They are

$$\begin{array}{rrrrrr} 1, & 1, & 1, & 1, & -1, & -1 \\ 1, & -1, & 1, & 1, & 1, & -1 \end{array} \tag{27}$$

and one seed sequence of period 3,

$$1, \quad -1, \quad 1, \quad 1, \quad -1, \quad 1.$$

Groups 2) and 3) present only *two* cases as shown in (27).

Finally, let us examine group 1), which consists of three seed sequences

$$1, \quad 1, \quad 1, \quad -1, \quad -1, \quad -1 \tag{28a}$$

$$1, \quad 1, \quad -1, \quad 1, \quad -1, \quad -1 \tag{28b}$$

$$-1, \quad -1, \quad 1, \quad -1, \quad 1, \quad 1. \tag{28c}$$

(28b) is the reflection of (28c) and one seed sequence of period 2

$$1, \quad -1, \quad 1, \quad -1, \quad 1, \quad -1.$$

It can now be concluded, from (26)–(28), that the number of cases to be investigated is *five*.

If the plant transfer function has an integrating factor $1/s$ (this is not at all uncommon), then the sum of the coefficients on the left-hand side of (2) is zero, that is

$$\sum_{l=0}^{p} a_{p-l} = 0. \tag{29}$$

Let $r=0$ in (8); one obtains

$$v_0 \sum_{l=0}^{p} a_{p-l} = g_0 \sum_{l=0}^{p} b_{p-l}. \tag{30}$$

Considering (29) and (30), one concludes that if

$$\sum_{l=0}^{p} b_{p-l} \neq 0,$$

then g_0 has to be zero, that is to say, the sum of the terms in the sequence to be investigated has to be zero. Under this condition, only *two* cases from (28) have to be considered.

Table II shows the seed sequences that have to be considered for a specified N. The coefficients of (17) are computed by using (5) for each sequence. As soon as the investigator normalizes his relay output levels to 1 and −1, they can be readily used in (8) and (9).

Tables I and II are universal in usage and can be easily expanded when it is necessary.

TABLE II

N	Seed Sequences	Coefficients of Expansion							
		g_0	g_1	g_2	g_3	g_4	h_1	h_2	h_3
2	1, −1	0	1						
4	1, 1, −1, −1	0	1	0			1		
6	1, 1, 1, −1, −1, −1	0	2/3	0	1/3		$2/\sqrt{3}$	0	
	1, 1, −1, 1, −1, −1	0	1/3	1	−1/3		$1/\sqrt{3}$	$1/\sqrt{3}$	
8	1, 1, 1, 1, −1, −1, −1, −1	0	1/2	0	1/2	0	$\frac{1}{2}(\sqrt{2}+1)$	0	$\frac{1}{2}(\sqrt{2}-1)$
	1, 1, −1, −1, 1, −1, 1, −1	0	$\sqrt{2}/4$	1/2	$-\sqrt{2}/4$	1/2	$\frac{1}{4}(\sqrt{2}-2)$	1/4	$\frac{1}{4}(\sqrt{2}+2)$
	1, 1, 1, −1, 1, −1, −1, −1	0	$\sqrt{2}/4$	1/2	$-\sqrt{2}/4$	1/2	$\frac{1}{4}(\sqrt{2}+2)$	1/4	$\frac{1}{4}(\sqrt{2}-2)$
	1, 1, −1, 1, 1, −1, −1, −1	0	0	1	0	0	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$
	1, 1, −1, 1, −1, −1, 1, −1	0	1/2	0	1/2	0	$\frac{1}{2}(\sqrt{2}-1)$	0	$\frac{1}{2}(\sqrt{2}+1)$

Discussion

A. R. Bergen (University of California, Berkeley). The authors present a method for determining whether a specified oscillation is possible in a relay sampled-data system. A particular periodic relay operating schedule is first assumed, and the steady-state sampled plant output is then computed; if this output causes the relay to operate as assumed, the oscillation is possible. Basically, the crux of the problem is to specify the steady-state sampled plant output when a periodic plant input is specified. The authors describe a method for accomplishing this; an alternate method, which is perhaps more direct, is described below. The method is entirely analogous to that used by Bass[1] for continuous relay systems.

To explain the method consider the authors' illustrative example, part 1). The output of the relay is assumed to alternate between -1 and $+1$ during successive sampling intervals. Then, assuming the authors' starting phase, the Laplace transform of the relay output is

$$F(s) = \frac{-1}{s} \frac{1 - 2z^{-1} + z^{-2}}{1 - z^{-2}} = -\frac{1}{s} \frac{1 - z^{-1}}{1 + z^{-1}} \tag{1}$$

where $z = e^{sT}$.

If oscillations in this mode are possible at all, plant initial conditions may be found so that the plant output assumes its steady-state behavior immediately, without a transient. In this case, and assuming a periodic oscillation, the Laplace transform of the sampled output is

$$C(z) = c_0 + c_1 z^{-1} + c_0 z^{-2} + c_1 z^{-3} + c_0 z^{-4} + \cdots$$

$$= \frac{c_0 + c_1 z^{-1}}{1 - z^{-2}} = \frac{c_0 + c_1 z^{-1}}{(1 - z^{-1})(1 + z^{-1})}. \tag{2}$$

Now if the response to the excitation given by (1) is an oscillation in the form of (2), with c_0 positive and c_1 negative, the relay will close as assumed and the oscillation is possible.

In computing $C(z)$ the initial plant conditions must be considered. Returning to the differential equation governing the plant

$$\ddot{c} + \dot{c} = f. \tag{3}$$

The Laplace transform is found to be

$$s(s + 1)C(s) = F(s) + (s + 1)c_0 + \dot{c}_0. \tag{4}$$

Inserting $F(s)$ from (1),

$$C(s) = \frac{-1}{s^2(s + 1)} \frac{1 - z^{-1}}{1 + z^{-1}} + \frac{c_0}{s} + \frac{\dot{c}_0}{s(s + 1)}. \tag{5}$$

Using a table of z transforms,

$$C(z) = \frac{-Tz^{-1}}{(1 - z^{-1})(1 + z^{-1})} + \frac{(1 - e^{-T})z^{-1}}{(1 + z^{-1})(1 - e^{-T}z^{-1})} + \frac{c_0}{1 - z^{-1}} + \frac{\dot{c}_0(1 - e^{-T})z^{-1}}{(1 - z^{-1})(1 - e^{-T}z^{-1})}. \tag{6}$$

The question is now this: can (6) be put into the form of (2), with c_0 positive and c_1 negative? If so, the oscillation is possible. Equating (2) and (6),

$$c_0 + c_1 z^{-1} = -Tz^{-1} + c_0(1 + z^{-1}) + \frac{(1 - e^{-T})z^{-1}}{1 - e^{-T}z^{-1}} [(1 + \dot{c}_0) + (1 - \dot{c}_0)z^{-1}], \tag{7}$$

a condition which is clearly impossible, unless the denominator of the third term on the right of (7) is cancelled. This occurs when

$$\dot{c}_0 = \frac{1 - e^{-T}}{1 + e^{-T}}, \tag{8}$$

and in this case,

$$c_0 + c_1 z^{-1} = c_0 + \left(c_0 - T + 2 \frac{1 - e^{-T}}{1 + e^{-T}}\right) z^{-1}. \tag{9}$$

The authors assume $T = 1$. Then

$$c_0 + c_1 z^{-1} = c_0 + (c_0 - 0.076)z^{-1}. \tag{10}$$

Now if $0 < c_0 < 0.076$, c_0 will be positive, c_1 will be negative, and the postulated oscillation is possible. This same condition is obtained by the authors.

The applicability of the method is obviously not restricted to the illustrative example. The general testing testing procedure remains the same. Some of the arbitrary initial conditions are used up in satisfying the requirement of the assumed periodicity. The remaining initial conditions are used in attempting to satisfy the requirements of sign of the c_i.

The method of the authors and that of this discussion share the advantages of being exact and of not being limited by dimensionality. Both methods, however, require an exhaustive investigation to reveal possible modes, and furnish necessary but not sufficient conditions for their existence.

Authors' Comments. The authors believe that the approach we present furnishes a simple and exhaustive

[1] R. W. Bass, "Equivalent linearization, nonlinear circuit synthesis and the stabilization and optimization of control systems," *Proc. Symp. on Nonlinear Circuit Analysis*, Interscience Publishers, Inc., New York, N. Y., vol. VI, pp. 163–198; April, 1956.

study of the periodic modes of a relay sampled-data system. The relay output schedule is not entirely assumed, but is based on the work by Izawa and Weaver. It is the authors' belief that this method, together with the Izawa and Weaver approach, will be a powerful tool in investigating the periodic modes of a relay sampled-data system.

If one cares to look carefully at the development of the art of sampled-data control systems, he will realize that it has partly been an effort to bring to the discrete domain the techniques useful in studying the continuous control systems. In linear system analysis, for example, the prevailing transformation methods, and certain processing techniques, all represent attempts in this direction. Correspondingly, in nonlinear system analysis, we have the phase-plane approach and describing function approach. But in doing this, one has to be aware of the fundamental differences between a continuous system and a discrete system. For instance, certain oscillations in a sampled-data relay system will not be found in a continuous system with the same plant and relay characteristics. The remark, "The method is entirely analogous to that used by Bass[1] for continuous relay systems," would be quite justified by casually glancing through the opening paragraphs of Bass's work, and, in the phrase, "The discovery of all periodic solutions of a given system of ordinary non-linear differential equations," substituting the word "difference" for "differential." But one can not ignore the basic difference between the nonlinear differential equation and a nonlinear difference equation. If one does examine in detail Bass's work and other more recent work[2] in Russia on the continuous relay system, he will realize the significant difference in analyzing a continuous system and a discrete system. It is the authors' opinion that Bergen's remark is irrelevent.

We find that Bergen's alternate method is in essence really "entirely analogous" to our method. We are fully aware of the usefulness of the z-transform technique, but we do not believe that its application in the present study will make it more direct, as asserted by the discussor.

Let us now look at this alternate method.

First of all, our starting phase in the illustrative example is $+1$, -1 instead of what Bergen "assumes" it to be.

[2] M. A. Aizerman and F. R. Gantmakher, "Determination of periodic modes in systems with piecewise-linear characteristics composed of segments parallel to two specified straight lines," *Automation and Remote Control*, vol. 18, pp. 105–120; February, 1957.

For convenience of comparison, we adopt the -1, 1 starting phase in the following discussion.

Bergen's Method	*Our Method*
1) The relay output $Y(s) = -\frac{1}{s}\frac{1-z^{-1}}{1+z^{-1}}$. One has to take the Laplace transform of a discontinuous function, and this needs manipulation to obtain the closed form. When N is a large number, this task is somewhat formidable.	$Y(n) = -\operatorname{Cos} n\pi$ n from $0 \to \infty$. This can be obtained by inspection in simple cases, or computation from (5), or referring to Table II in the Appendix.
2) The system output at the sampling instant: $c(z) = c_0 + c_1 z^{-1} + \cdots = \frac{c_0 + c_1 z^{-1}}{(1-z^{-1})(1+z^{-1})}$. Manipulation is needed to obtain the closed form.	$c(n) = v_0 + v_1 \operatorname{Cos} \pi n$. No manipulation is needed, just write down the expression from (6).
3) The plant equation: Write down the differential equation from the transfer function of the given plant. Take the Laplace transform. This step is to bring in one more unknown, $\dot{c}_0$ in this case, more in higher order systems. $s(s+1)(s) = Y(s) + (s+1)c_0 + \dot{c}_0$.	From the transfer function of the given plant, the difference equation is obtained by direct integration or with the help of a z-transform table. $c_{n+2} - 1.368c_{n+p} + 0.368c = 0.368y_{n+1} + 0.264y_n$. No additional unknown is introduced.
4) Take the z-transform and obtain a bulky equation for $C(z)$. This could be more cumbersome with a large period of oscillation.	No corresponding step necessary.
5) Equate the expression in step 2) to the long expression in step 6) and arrange terms.	Substitute coefficients obtained from steps 1), 2), and 4) to 8) to obtain two linear equations in terms of v_0 and v_1 which can be easily solved.
6) Inspect the equation in step 5), attempt to find suitable condition for $\dot{c}_0$. It should be remarked here for a higher order system there seems to be no general rule to follow to eliminate these unnecessarily introduced unknowns.	No corresponding step necessary.
7) Find the range for c_0 and deduce c_1 by keeping the validity of the result in mind.	Check the validity by merely inspecting the signs of c_0 and c_1.

After the above step by step comparison, Bergen's claim that his method is perhaps more direct clearly needs qualification.

We are grateful to Professor Bergen for his discussion.

Analysis of Pulse Duration Sampled-Data Systems with Linear Elements*

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Summary—**This paper deals with the application of *z*-transform techniques to the analysis of sampled-data systems in which signals appear in *pulse duration* modulated form. The characteristics of pulse duration sampled data are first briefly described. An analytical means for studying the transient response and stability of systems which use this kind of data is then presented. Finally, a comparison is shown of analytical results with tests on several representative experimental systems. A short discussion of the relative advantages and disadvantages of pulse duration modulation in control systems is also included.**

Introduction

THE STUDIES of sampled-data systems made up to now have ordinarily assumed that sampling is a kind of *pulse amplitude* modulation process. The output of the sampler is usually thought of as a sequence of equally spaced, equal duration pulses whose amplitudes are proportional to the input to the sampler at the sampling instants (constant sampling frequency and constant pulse width). There are, however, other ways of representing sampled intelligence by a modulation process. One of these requires that the sampler produce an output pulse train made up of equally spaced, constant amplitude pulses, whose individual *pulse durations* vary as a function of the input to the sampler at the sampling instants. In contrast to the former, this is a pulse duration modulation process. These two types of sampled data illustrated in Fig. 1 will be referred to in this paper by the terms pulse amplitude sampled data (PASD) and pulse duration sampled data (PDSD).[1]

The classical example of a system which employs PDSD illustrated in Fig. 2, was described by Gouy approximately sixty years ago.[2] This is a temperature regulating system which consists of an oven heated by a resistive heating element, a source of electrical potential, a relay, a mercury thermometer, and an electrical contact mounted on a motor driven eccentric. Whenever the electrical contact becomes immersed in the mercury, current flows in the coil of the relay which interrupts the heating current. Since the contact is periodically dipped into the mercury, the heating current consists of a sequence of pulses. These pulses have equal amplitudes, but their widths depend upon the time that the contact is immersed in the mercury. The time that the contact is immersed in the mercury depends upon the level of the mercury in the thermometer, and the level of the mercury depends upon the temperature of the oven. Thus the current pulse widths are varied in proportion to the temperature of the oven, and feedback control of the oven temperature is attained.

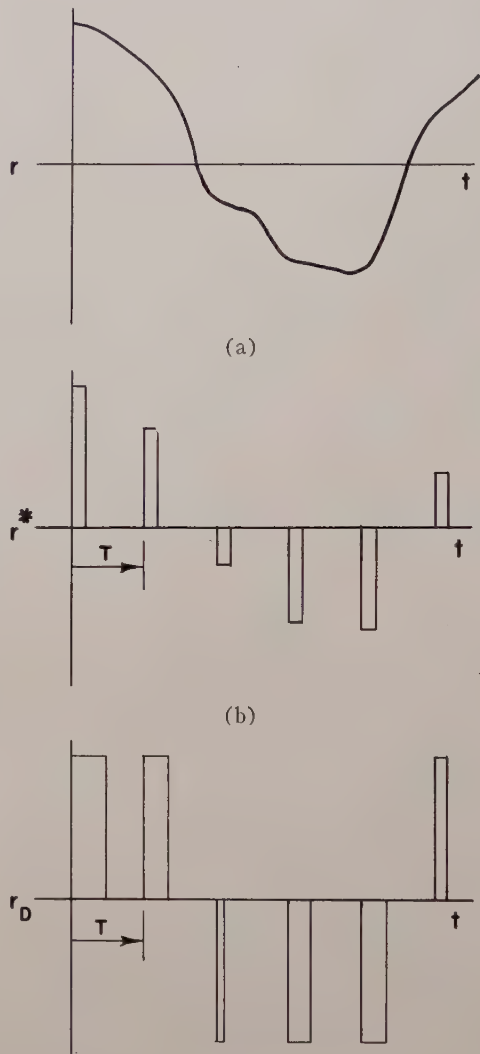

Fig. 1—Comparison of sampled data. (a) Input; (b) PASD output; (c) PDSD output.

One of the reasons for the past emphasis on PASD is that the mathematical methods for analyzing control systems based on this type of data have been extensively developed. A pulse amplitude sampler is essentially a linear device in the sense that inputs and outputs are governed by the superposition principle. This is not the case with a pulse duration sampler, and as a consequence

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† Sperry Phoenix Co., a division of Sperry Rand Corp., Phoenix, Ariz.

[1] H. S. Black, "Modulation Theory," D. Van Nostrand Co., Inc., New York, N. Y.; 1953.

[2] M. Gouy, "On a constant temperature oven," *J. Physique*, vol. 6, ser. 3, pp. 479–483; 1897.

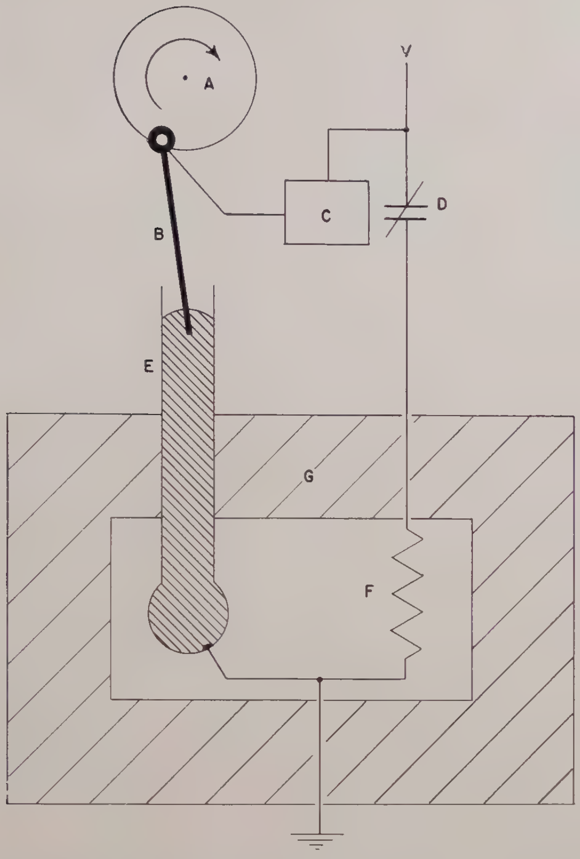

Fig. 2—Gouy temperature regulator. A = motor driven eccentric, B = electrical contact, C = relay coil, D = normally closed contacts, E = mercury thermometer, F = resistive heating element, G = oven.

the development of mathematical means for analyzing PDSD systems has been hindered. On the other hand, PDSD systems have a number of advantages which recommend their use. This paper shows how the Principle of Equivalent Areas[3] may be applied to the mathematical analysis of PDSD systems with linear continuous elements.

The Characteristics of Pulse Duration Sampled Data

The PDSD described in this paper is generalized somewhat to include the possibility of a negative sampler output as well as positive and zero outputs. The polarity of the sampler output depends upon the polarity of the input to the sampler. Furthermore, it is assumed that the sampler output pulses all have their leading edges at the sampling instants $t = nT$. The locations of the trailing edges of the pulses are assumed to be a linear function of the input to the sampler at the sampling instants. In the original Gouy temperature regulator the output pulses are symmetrical with respect to the sampling instants, and consequently differ slightly from the assumptions which we have made for mathematical convenience. However, since the temperature change is assumed small during a sampling period this difference is of little consequence.

Consider Fig. 1. If the input to a pulse duration sampler is $r(t)$, then the sampled output $r_D(t)$ is given by the following expression:

$$r_D(t)$$

$$= E \sum_{n=0}^{\infty} \operatorname{sign} r(nT)[u_{-1}(t - nT) - u_{-1}(t - nT - h_n)], \quad (1)$$

where h_n, the variable pulse duration, is equal to the following

$$h_n = a|r(nT)| \qquad a|r(nT)| \leq T$$
$$h_n = T \qquad a|r(nT)| > T.$$

It is apparent at the very outset that while a pulse amplitude sampler is a linear element in the sense that inputs and outputs are governed by the superposition principle, a pulse duration sampler is a nonlinear element. Two types of nonlinearity are present. First, superposition does not apply; that is, the sampled sum of two inputs does not equal the sum of two sampled outputs corresponding to these inputs. Second, the sampler output saturates when $a|r(nT)| = T$ since obviously the pulse duration cannot exceed the sampling period. In the following analysis we will always assume that the input to the sampler is not greater than the value which causes saturation. In other words, we will restrict the sampler operation to a quasi-linear unsaturated region.

Systems with PASD usually depend upon the use of a hold circuit to convert the sampler output pulse train into a staircase type signal. This is necessary for two reasons. First, converting the pulsed input to a staircase input allows more energy to be delivered to the system during a sampling period. If one increases the pulse amplitude instead, one runs into the problem of saturation. Second, this reduces the amount of ripple in the output because the hold circuit acts as a low pass filter on the pulsed signal. Systems with PDSD must depend on the system elements themselves to perform the filtering function. These systems must include some energy storage device such as a motor inertia or an oven heat capacity which will smooth out the effects of the pulsed nature of the PDSD input. To provide adequate filtering, the following discussion and analysis consequently assumes that a pulse duration sampler always works into a combination of system elements which includes a low pass element having a largest time constant at least twice the sampling period (or if the roots are imaginary, the reciprocal of the real part of the largest root is at least twice as great as the sampling period).

There is a great deal of similarity between the output of a pulse duration sampler and the output of an on-off relay controller. Both outputs are pulsed and may assume only the amplitudes $+E$, zero, and $-E$. The principal difference between the outputs is that the sampling period of the pulse duration sampler is fixed,

[3] R. E. Andeen, "The principle of equivalent areas." (To be published in *Trans. AIEE.*)

while both the pulse width and pulse repetition rate of the on-off relay controller depend upon the input to the controller. This similarity is not exploited in the following.

The following definitions will be important in the analysis of PDSD systems. We assume that the absolute value of the input to the pulse duration sampler, $|r(t)|$, is bounded and $R_{\max}$ is equal to the largest value of $|r(t)|$ which we measure. The maximum pulse width $aR_{\max}$ by assumption is less than, or equal to, the sampling period T. The modulation ratio, β, is defined as the ratio of $aR_{\max}$ to the sampling period T,

$$\beta = \frac{aR_{\max}}{T}. \tag{2}$$

β must lie between zero and one. The filter ratio, α, is defined as the ratio of the sampling period to the largest time constant of the system,

$$\alpha = \frac{T}{T_{\max}}. \tag{3}$$

By assumption, α must lie between zero and one half.

Analysis of Linear Systems

This analysis is based on the principle of equivalent areas (Appendix) and the z-transform method. The principle of equivalent areas expresses the conditions under which two input signals to a dynamic element produce essentially the same output. According to this principle, two input signals are dynamically equivalent if their integrals evaluated over corresponding sampling intervals are equal, and the sampling frequency determined by this interval is at least twice the highest significant frequency in the frequency spectrum of the dynamic element. Expressed mathematically, two input signals are equivalent for suitably small T if

$$I_n = \int_{(n-1)T}^{nT} r(t)dt = \int_{(n-1)T}^{nT} r'(t)dt = I_n'$$

for all values of n sufficient to cover the time interval of interest.

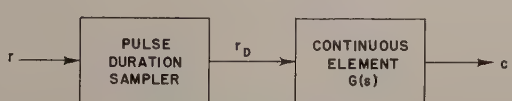

Fig. 3—System with pulse duration sampler.

Consider Fig. 3 and assume that $r(t)$ is some input control signal. $r(t)$ is sampled by the pulse duration sampler. The output $r_D(t)$ is PDSD and is the input to the continuous linear element $G(s)$. We are concerned with determining the behavior of the output $c(t)$ when $r(t)$ and the system parameters are known. To do this the pulse duration sampler is replaced by an equivalent circuit comprised of a pulse amplitude sampler in cascade with a hold circuit described by a transfer function $H(s)$. Restricting attention to the sampling instants, the z-transform method is applied to the linearized system to determine the equivalent z-transform of the overall PDSD system. Transient response and stability boundaries can then be investigated by z-transform techniques in the usual way.

The replacement of the pulse duration sampler by an equivalent circuit is accomplished as follows. Refer to Fig. 4. This figure shows some arbitrary sampler input $r(t)$ and the corresponding PDSD output $r_D(t)$. A PASD signal $r_D'(t)$ is constructed from $r_D(t)$ according to the Principle of Equivalent Areas. T is chosen as an interval over which area must be conserved. The areas of corresponding pulses of $r_D(t)$ and $r_D'(t)$ are required to be identical. Under these conditions the equivalent PASD is related to $r(t)$ by the following expression,

$$r_D'(t) = \frac{E}{R_{\max}} \sum_{n=0}^{\infty} r(nT) \cdot [u_{-1}(t - nT) - u_{-1}(t - nT - \beta T)]. \tag{4}$$

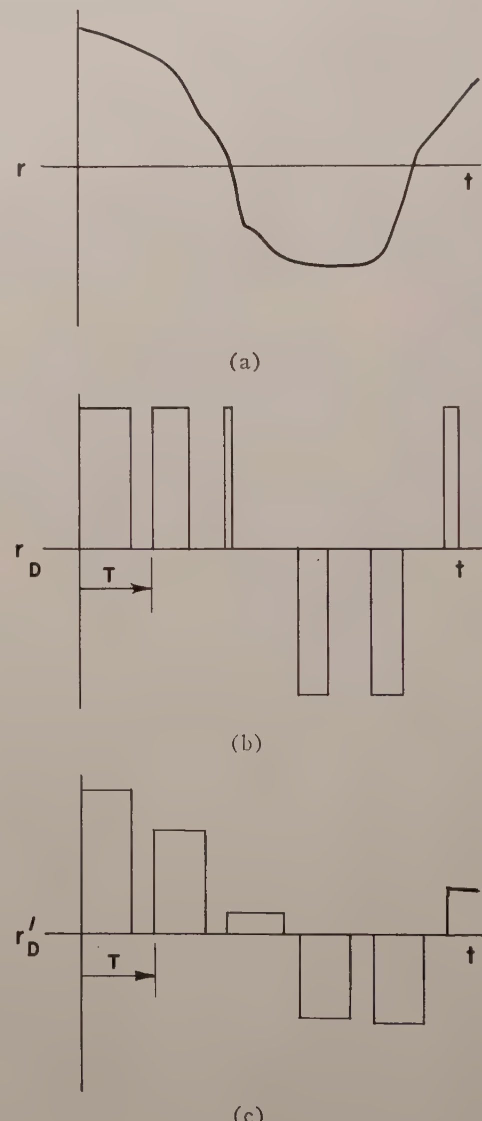

Fig. 4—Equivalent sampled data. (a) Input signal; (b) PDSD signal; (c) equivalent PASD signal

The Laplace transform of (4) is

$$R_d'(s) = \frac{1 - e^{-\beta T s}}{s} \frac{E}{R_{\max}} \sum_{n=0}^{\infty} r(nT) e^{-nTs}. \tag{5}$$

Eq. (5) can be looked on as representing an amplitude sampler in cascade with a hold circuit with the characteristic

$$H(s) = \frac{1 - e^{-\beta T s}}{s} \frac{E}{R_{\max}}. \tag{6}$$

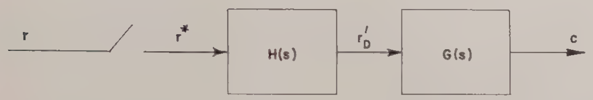

Fig. 5—Equivalent circuit for system with pulse duration sampler.

Fig. 5 thus represents a linearized equivalent circuit corresponding to Fig. 4 which includes an amplitude sampler in cascade with a hold circuit to represent the pulse duration sampler. One must remember that the validity of this equivalent circuit depends upon $G(s)$. $G(s)$ must act as a low pass filter whose dominant time constant is at least twice the sampling period ($0 < \alpha < \frac{1}{2}$). This, as pointed out previously, is usually the case in a practical system where filtering action is necessary to attenuate ripple in the output.

The over-all equivalent z-transform c^*/r^* which represents the system comprised of a pulse duration sampler in cascade with $G(s)$ is easily determined for any particular $G(s)$. Consider the following example. Assume that $G(s)$ is a simple lag network whose transfer function is

$$G(s) = \frac{A}{s + A}. \tag{7}$$

The relations in Fig. 5 are represented mathematically by

$$C(s) = H(s)G(s)R^*(s) \tag{8}$$

$$C^*(z) = \mathrm{Z}[H(s)G(s)]R^*(z). \tag{9}$$

The z-transform of $H(s)G(s)$ is evaluated by complex integration as shown by Jury.[4]

$$\mathrm{Z}[H(s)G(s)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} H(p)G(p) \frac{1}{1 - e^{-T(s-p)}} dp. \tag{10}$$

$$\mathrm{Z}[H(s)G(s)] = \frac{E}{R_{\max}} \left\{ \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{1 - e^{-\beta T p}}{p} \frac{A}{(p + A)} \cdot \frac{dp}{1 - e^{-T(s-p)}} \right\}. \tag{11}$$

[4] E. I. Jury, "Analysis and synthesis of sampled data control systems," *Trans. AIEE*, vol. 73, pp. 332–346; September, 1954.

Evaluating by residues we obtain,

$$\mathrm{Z}[H(s)G(s)] = \frac{E}{R_{\max}} \left\{ \left[\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-AT} z^{-1}} \right] - z^{-1} \left[\frac{1}{1 - z^{-1}} \frac{e^{-AT(1-\beta)}}{1 - e^{-AT} z^{-1}} \right] \right\}. \tag{12}$$

The following equation results from combining terms and substituting $\alpha = AT$:

$$\mathrm{Z}[H(s)G(s)] = \frac{E}{R_{\max}} \left\{ 1 - \frac{1}{1 - e^{-\alpha} z^{-1}} + \frac{e^{-\alpha + \alpha\beta} z^{-1}}{1 - e^{-\alpha} z^{-1}} \right\}. \tag{13}$$

Eq. (13) can be reduced to the following,

$$\mathrm{Z}[H(s)G(s)] = \frac{e^{-\alpha}(e^{\alpha\beta} - 1) z^{-1}}{1 - e^{-\alpha} z^{-1}} \frac{E}{R_{\max}}. \tag{14}$$

Thus the result is

$$\frac{C^*(z)}{R^*(z)} = \frac{E}{R_{\max}} \frac{e^{-\alpha}(e^{\alpha\beta} - 1) z^{-1}}{1 - e^{-\alpha} z^{-1}}. \tag{15}$$

The results of similar derivations for some PDSD systems with different continuous elements are shown in Table I. The reader can readily extend this table to include any $G(s)$ of special interest by following the procedure described above.

TABLE I

List of Equivalent PDSD System z-Transforms

$G(s)$	Equivalent PDSD System z-Transform C^*/R^*
$\frac{A}{s + A}$	$\frac{E}{R_{\max}} \left[\frac{e^{-\alpha}(e^{\alpha\beta} - 1) z^{-1}}{1 - e^{-\alpha} z^{-1}} \right]$
$\frac{AB}{(s + A)(s + B)}$	$\frac{E}{R_{\max}} \left[\left(\frac{\alpha}{\alpha - \gamma} \right) \frac{(e^{\gamma\beta} - 1) e^{-\gamma} z^{-1}}{1 - e^{-\gamma} z^{-1}} + \left(\frac{\gamma}{\gamma - \alpha} \right) \frac{(e^{\alpha\beta} - 1) e^{-\alpha} z^{-1}}{1 - e^{-\alpha} z^{-1}} \right]$
$\frac{A}{s(s + A)}$	$\frac{E}{R_{\max}} \left[\frac{\beta z^{-1}}{1 - z^{-1}} \quad \frac{T e^{-\alpha}(1 - e^{\alpha\beta}) z^{-1}}{(1 - e^{-\alpha} z^{-1})} \right]$

Note:

$$\alpha = AT$$
$$\gamma = BT$$
$$\beta = \frac{a R_{\max}}{T}.$$

Advantages and Disadvantages of Using Pulse Duration Sampled Data

Some of the advantages of using PDSD in a feedback control system are the following:

1) PDSD is less susceptible to the influence of noise which manifests itself by random amplitude variations since the information contained by an individual pulse resides in the length of the pulse. Random amplitude variations can be eliminated by clipping.

2) The output stage of the pulse duration sampler can be a simple relay circuit. Thus very high power gains can be realized with little equipment complexity and relatively low cost.
3) In the presence of sticking friction or backlash it is more effective to apply a large torque for a short time than a smaller torque for a relatively longer time. A system based on the use of PDSD to drive an output motor against a load with friction and backlash applies full torque to the load for a greater or shorter time depending upon the input to the sampler. This is consequently more effective than a system based on the use of PASD (with a hold circuit) which applies a torque proportional to the input to the sampler for an entire sampling period. In order to obtain an acceptable torque gradient (pound-foot/volt of error) with the system based on PASD (or for that matter, also systems using continuous data) an output motor is often used which has a much higher rated torque than that which is actually required for maximum load. This is inefficient, more costly, and introduces the problem of torque limiting.
4) Torque limiting is particularly important in aircraft applications where exceeding the structural limits of the aircraft would be disastrous. When PDSD is used to drive the output motor, torque limiting is easily accomplished since only one value of torque is ever applied. The associated circuitry can easily be designed to be fail-safe.
5) Compared to an on-off regulator, the pulse duration sampling regulator has the following advantage. With the on-off regulator the steady-state ripple in the output is a function of the gain of the system and the dead band of the relays. The frequency of the ripple may change with system gain, but the amplitude is fixed by the dead band. With the pulse duration sampling regulator, the frequency of the ripple is a constant which is completely under the control of the designer. The amplitude of the ripple is still a function of the gain of the system, however, by increasing the sampling frequency and reducing the system gain at the same time to maintain a constant ratio of on time to total time, the ripple can be reduced to any desired level.
6) No hold circuit is required.

Some of the disadvantages of using PDSD are the following:

1) PDSD can only be used where low pass filter elements provide adequate filtering to reduce the ripple due to the pulsed input to an acceptable level.
2) A pulsed input and consequent pulsating torque may cause greater heating and mechanical stresses in an output motor and associated load elements. This can be reduced somewhat by introducing additional filtering.

Experimental Results

A series of tests were made to obtain an experimental confirmation of the theoretical results given in Table I. A pulse duration sampler and various linear systems were constructed with electronic and relay elements and an analog computer as described in the Appendix, and were subjected to step and sine-squared pulse

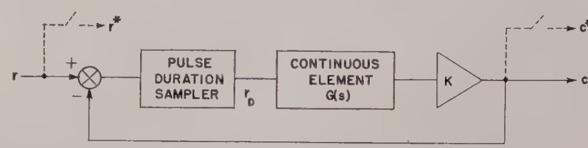

Fig. 6—PDSD system with feedback.

transients. Figs. 3 and 6 illustrate the configurations of the open-loop and feedback systems tested. Several continuous elements were considered:

System I

$$G(s) = \frac{A}{(s + A)}.$$

System II

$$G(s) = \frac{AB}{(s + A)(s + B)}.$$

System III

$$G(s = \frac{A}{s(s + A)}.$$

Three combinaations of parameters were studied,

Case a) $\alpha = \frac{1}{2}$
$\beta = \frac{1}{2}$
$T = 8$ seconds

Case b) $\alpha = \frac{1}{4}$
$\beta = \frac{3}{4}$
$T = 8$ seconds

Case c) $\alpha = \frac{1}{8}$
$\beta = \frac{1}{2}$
$T = 8$ seconds.

The theoretical responses which would be obtained from the open-loop and feedback systems illustrated in Figs. 3 and 6 for the given transient inputs were also calculated by means of the equations in Table I and the usual z-transform methods. The experimental and theoretical results for Case b) are compared in Figs. 7 and 8 for the open-loop and feedback cases respectively. The results for Cases a) and c), not shown here, were similar. From these figures it appears that in both the open-loop and feedback situations the equations of Table I were able to predict the performance of the PDSD systems studied to a degree of approximation satisfactory for practical purposes.

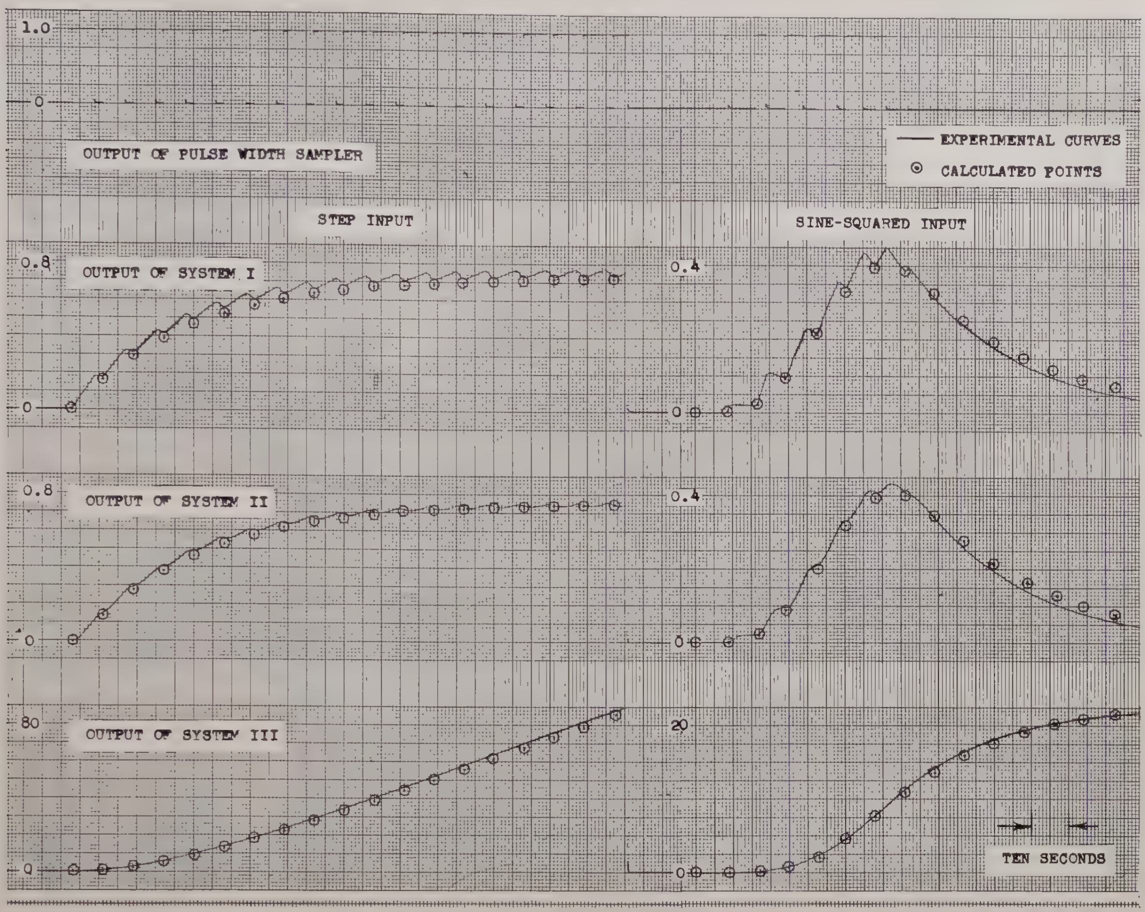

Fig. 7—Comparison of experimental and calculated transient responses of open-loop PDSD systems. Case b): $\alpha=\frac{1}{4}$, $\beta=\frac{3}{4}$, $T=8$ seconds.

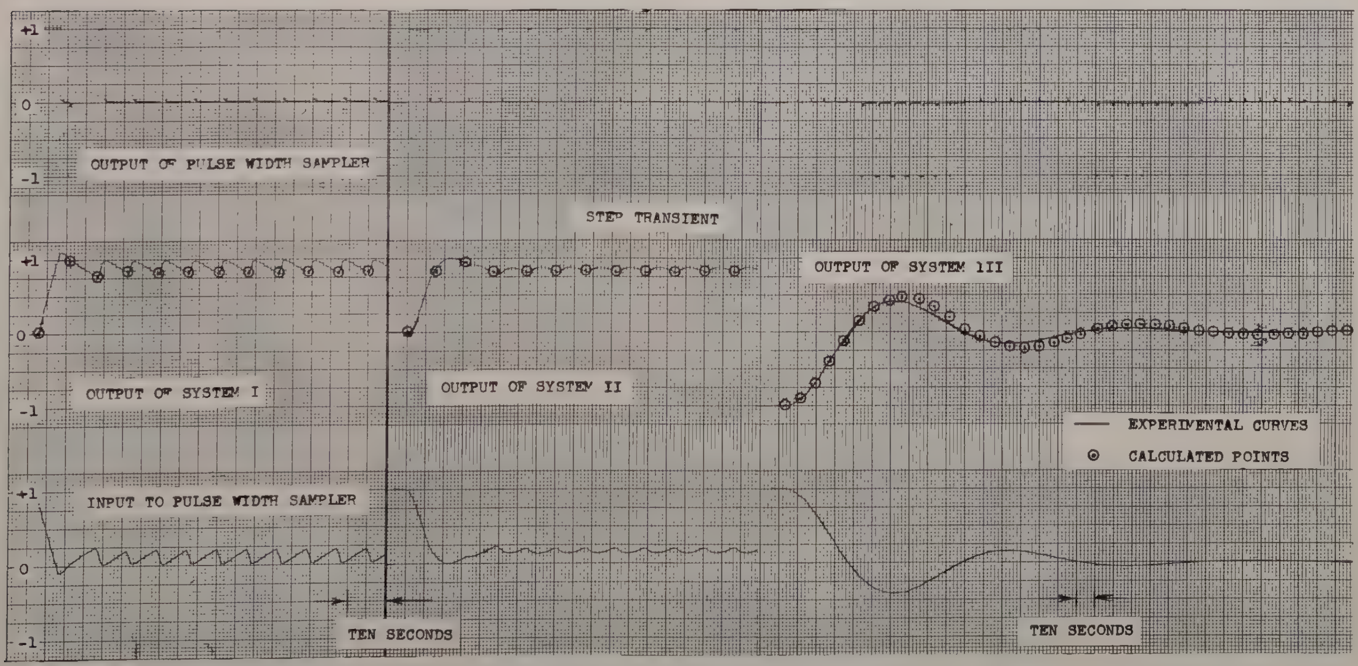

Fig. 8—Comparison of experimental and calculated transient responses of feedback PDSD systems. Case b): $\alpha=\frac{1}{4}$, $\beta=\frac{3}{4}$, $T=8$ seconds.

The stability boundaries of the PDSD feedback systems illustrated by Fig. 6 were also calculated from the equations of Table I and compared with experimentally determined stability boundaries. The results are summarized in Table II. It is apparent that the agreement between the experimental and calculated results is good.

TABLE II
GAIN AT STABILITY BOUNDARY

System	Case	K_0 Calculated	K_0 Experimental
I	a)	9.34	9.00
	b)	11.1	9.95
	c)	33.1	29.3
II	a)	10.47	9.25
	b)	16.1	15.0
	c)	41.0	33.8
III	a)	1.19	1.20
	b)	0.474	0.55
	c)	1.045	1.10

Conclusions

An approximate method of analyzing PDSD systems with linear elements can be based on the z-transform method if the PDSD signal is replaced by an equivalent PASD signal. This replacement is based on the principle of equivalent areas which requires that the areas of the corresponding pulses of the actual and equivalent sampled signals to be equal. This type of linearized analysis is based on two assumptions. First, the operation of the pulse duration sampler is restricted to a quasi-linear region where input variations can be translated into pulse duration variations. Second, it is assumed that the continuous elements following the pulse duration sampler include a low pass element which has a time constant which is at least twice as great as the sampling period. This condition is not very restrictive, for any practical system requires filtering to reduce the effects of the pulsed signal.

When we replace the PDSD signal by an equivalent PASD signal we find that a pulse duration sampler can be represented by an equivalent circuit consisting of an ideal amplitude sampler and a special hold circuit whose parameters are a function of the modulation ratio and the maximum value of the input. Based on this equivalent circuit the system z-transforms of a pulse duration sampler in combination with several different continuous elements were derived and are shown in Table I. A series of tests which were designed to obtain an experimental confirmation of these equations gave satisfactory results. We conclude that the method of studying the behavior of PDSD systems described here, subject to the above assumptions, will give results satisfactory for practical engineering calculations.

The success of this approximate method of analysis suggests that digital or discrete compensating elements could be employed to improve the transient response or other characteristics of PDSD systems. A large body of literature already exists which treats the application of digital or discrete compensation to PASD systems. Much of this can probably be carried over to PDSD systems. Mixed PASD and PDSD systems seem feasible.

Appendix I

Principle of Equivalent Areas

The *principle of equivalent areas*[3] is an approximation expressing the conditions under which two inputs to a dynamic element produce essentially the same output. The derivation starts with the convolution integral relating the output $c(t)$, input $r(t)$, and the impulsive response of the dynamic element $h(t)$,

$$c(\tau) = \int_{-\infty}^{\tau} h(\tau - t)r(t)dt.$$

We replace $h(t)$ by a staircase approximation $h_p(t)$ on the basis that $h_p(t)$ changes value at a sampling interval short enough so that the frequency spectra $H(j\omega)$ and $H_p(j\omega)$ are similar. A sampling frequency at least twice the highest significant frequency of $H(j\omega)$ will ordinarily accomplish this. Assume that τ is an integral number of sampling periods,

$$\tau = mT \qquad m = 0, 1, 2, 3 \cdots .$$

Then we can write,

$$c(mT) = \int_{-\infty}^{mT} h_p(mT - t)r(t)dt.$$

Furthermore the integral from $-\infty$ to mT can be broken up into a sum of integrals over the intervals $(n-1)T$ to nT where n goes from $-\infty$ to m,

$$c(mT) = \sum_{n=-\infty}^{m} \left[\int_{(n-1)T}^{nT} h_p(mT - t)r(t)dt \right].$$

However, during the interval $(n-1)T$ to nT, $h_p(mT-t)$ is constant and equal to $h_p(mT-nT)$. Consequently we obtain

$$c(mT) = \sum_{n=-\infty}^{m} \left[\int_{(n-1)T}^{nT} h_p(mT - nT)r(t)dt \right]$$

$$c(mT) = \sum_{n=-\infty}^{m} h_p(mT - nT) \int_{(n-1)T}^{nT} r(t)dt.$$

Now define $I(nT)$ by

$$I(nT) = \int_{(n-1)T}^{nT} r(t)dt.$$

Then we obtain our final result,

$$c(mT) = \sum_{n=-\infty}^{m} h_p(mT - nT)I(nT).$$

Appendix II

Experimental Equipment

The PDSD systems represented by Figs. 3 and 6

were set up in the laboratory using the following equipment:

1) low frequency square wave generator,
2) electronic analog computer,
3) 4-channel recorder,
4) 300-volt dc power supplies,
5) (amplitude) sampler and zero-order hold unit,
6) pulse duration relay unit.

The first four items are general laboratory type equipment, and the last two items were specially built for sampled-data studies.

Sampler and Zero-Order Hold Unit

This device operates to convert a continuous input voltage into a staircase output voltage which is equivalent to the output which one would obtain by the process of sampling and holding. The basic performance characteristics of this device are listed below:

range of sampling frequencies—0.20 to 2000 cps,
attenuation—approximately 0.84,
maximum voltage of input signal—± 25 volts peak,
phase shift—less than 10 degrees at highest frequency of the frequency selector setting.

The operation of the *S-H* unit is based on the principle shown in Fig. 9, but the switching is done electronically. The switches shown in Fig. 9 operate once each sampling interval. While one capacitor is charging up to the amplitude of the input, the other capacitor is disconnected from the input and supplies the output voltage to the load. A buffer amplifier is required to prevent discharging the capacitor connected to the load. A complete description of the electronic circuit, and details of operation and calibration are given by the author.[5]

Pulse Duration Relay Unit

The relay unit consists of five channels, each including an electronic amplifier which operates a sensitive relay. When a positive voltage is applied to the channel input the relay operates; for a negative voltage the relay drops out. Because of the high amplifier gain the differential voltage between pull-in and drop-out is insignificant.

PDSD System Operation

In setting up the systems described by Figs. 3 and 6 the continuous elements were simulated by the analog computer. A pulse duration sampler was constructed from the *S-H* unit, the relay unit, and an integrator and an amplifier of the analog computer. The operation of this simulated pulse duration sampler was as follows. The continuous input to the pulse duration sampler was applied to the *S-H* unit which converted this continuous voltage to a staircase type voltage. The output voltage of the *S-H* unit was applied to the input of an analog computer amplifier, which is designated *A*1 in Fig. 10. The input to the integrator was a constant voltage, and its output was also connected to the input of amplifier *A*1, but with a polarity opposite to that of the *S-H* unit output voltage. At the beginning of each sampling period the integrator was reset to zero output voltage, so that its output appeared as a sawtoothed voltage. A relay operated by the output voltage of amplifier *A*1 completed the pulse duration sampler. When the relay operated it applied the constant voltage $+E$ to the continuous element which followed it in the system. At the beginning of each sampling period the relay would first operate, applying $+E$ to the continuous element, because the polarity of the *S-H* unit output was the proper polarity to cause operation and the integrator output was zero. During the sampling interval the output of the integrator would increase and when it exceeded the *S-H* unit output the relay would drop out. It is clear that the time interval required for the integrator output to build up to the value of the *S-H* unit output was directly proportional to the magnitude of the output of the *S-H* unit. Thus the duration of the $+E$ output from the relay was made proportional to the input to the simulated pulse duration sampler at the sampling instants. Additional switching was arranged to take care of changing the polarity of the output and the input voltage to the integrator when the continuous input to the simulated pulse duration sampler changed polarity. Fig. 10 is simplified by showing the connections for operation with only one polarity of the input $r(t)$.

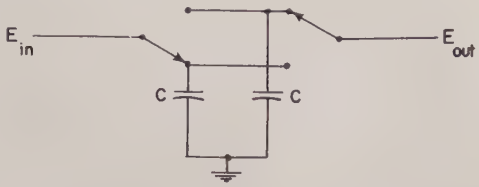

Fig. 9—Principle of operation of sampler and zero-order hold unit.

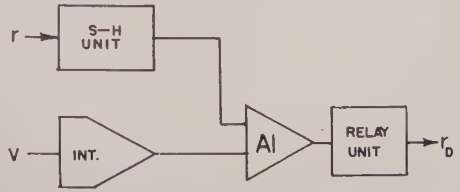

Fig. 10—Simplified block diagram of simulated pulse duration sampler.

Acknowledgment

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[5] R. E. Andeen, "A wide-range sampler with zero-order hold," *Control Engrg.*, vol. 6, pp. 149–151; May., 1959.

The Analysis of Cross-Coupling Effects on the Stability of Two-Dimensional, Orthogonal, Feedback Control Systems*

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Summary—The analysis of two-dimensional cross-coupled feedback control systems is studied because many such systems are in existence, and methods for their analysis and criteria for measuring their performance have had only limited attention in the technical literature [1]–[6]. Previous literature has generally been in the area of matrix analysis of multidimensional or multiloop systems, and has not treated the handling of poles and zeros in the complex s-plane or cross-coupling, as they are treated in this paper.

An algebraic model is developed for a generalized system, using the assumption that all components of the system, including the cross-coupling transfer function, may be represented by expressions in the complex frequency domain which have real-time domain equivalents describable by linear differential equations.

The stability of the two-dimensional, cross-coupled, feedback control system is studied by complex-plane methods, and a method is developed for determining whether a system is stable or not. A technique is also indicated to show how an unstable system may be made stable by the introduction of appropriate cross-coupling.

I. Introduction

Many two-dimensional, orthogonal, feedback control systems are in existence today. An excellent example of such a system is an automatically tracking radar system in which the antenna is continually directed in each of two orthogonal planes of action so as to minimize an error signal. The purpose of this system is to keep the antenna pointed towards the target, and thus establish the vector to the target from the point of observation.

In two-dimensional systems, cross-coupling occurs between the two orthogonal systems in many instances, because these systems share a common facility, such as a power supply or radio transmission link. The normal procedure for the analysis of two-dimensional systems has been to assume complete independence of the two systems, and to analyze each system independently. This paper discusses analytically the effect of cross-coupling on the stability of two-dimensional systems. It also shows that the introduction of cross-coupling into an unstable system can make the system stable.

II. Algebraic Model for Analysis

To establish a general framework for the analysis of a feedback control system it is necessary to develop an algebraic model.

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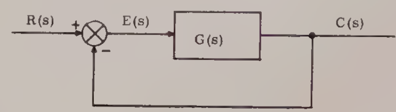
A one-dimensional feedback control system is generally represented by the following diagram,



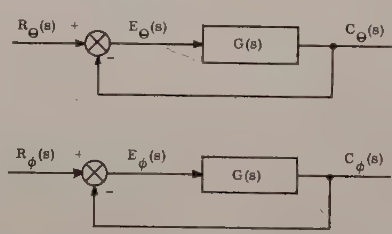
where s is the complex Laplace operator, $R(s)$ is the transform of the reference input function and $C(s)$ the controlled output function. $C(s)$ is to follow $R(s)$ as nearly as possible. $R(s)$ and $C(s)$ are compared continuously. $E(s)$ is the error function obtained by subtracting $C(s)$ from $R(s)$, and $C(s)$ is determined by $E(s)$ operating through a physical device whose transfer function is $G(s)$. An implicit assumption is that each term involved may be represented in real time by a linear differential equation, which is transformable from the time domain to the complex frequency (s) domain by the Laplace transformation,

$$F(s) = L[f(t)] = \int_0^\infty f(t)\epsilon^{-st}dt. \tag{1}$$

The total transfer function for the general one-dimensional system is usually expressed as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}. \tag{2}$$

Symmetrical two-dimensional, orthogonal systems containing no cross-coupling may generally be depicted by two such diagrams as,

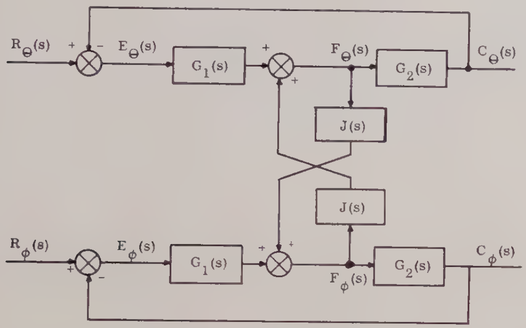


where θ and ϕ denote the two orthogonal planes of action. Thus, the total transfer function may be described by two equations,

$$\frac{C_\theta}{R_\theta} = \frac{G}{1+G} \tag{3}$$

$$\frac{C_\phi}{R_\phi} = \frac{G}{1+G}. \tag{4}$$

To depict linear cross-coupling in a more general fashion for a two-dimensional system, a general diagram for the total system may be represented as:



where $G(s)$ is now composed of $G_1(s)$ and $G_2(s)$ operating in series. The signal present at the output of $G_1(s)$ is summed with a signal from the orthogonal channel. The summed signal is the input to $G_2(s)$, and is also operated on by $J(s)$ to represent the signal cross-coupled between the two orthogonal systems. Algebraically, the following relations exist:

$$E_\theta = R_\theta - C_\theta \tag{5}$$

$$F_\theta = E_\theta G_1 + F_\phi J \tag{6}$$

$$C_\theta = F_\theta G_2 \tag{7}$$

$$E_\phi = R_\phi - C_\phi \tag{8}$$

$$F_\phi = E_\phi G_1 + F_\theta J \tag{9}$$

$$C_\phi = F_\phi G_2. \tag{10}$$

These six equations may be solved to eliminate F_θ, F_ϕ, E_θ, and E_ϕ. This solution provides the total transfer function for the system in two equations,

$$\frac{C_\theta}{R_\theta} = \frac{G}{(1+G)^2 - J^2}\left[1 + G + \frac{R_\phi}{R_\theta} J\right] \tag{11}$$

$$\frac{C_\phi}{R_\phi} = \frac{G}{(1+G)^2 - J^2}\left[1 + G + \frac{R_\theta}{R_\phi} J\right]. \tag{12}$$

These equations may be expressed in matrix form as,

$$\begin{bmatrix} C_\theta \\ C_\phi \end{bmatrix} = \begin{bmatrix} 1+G & J \\ J & 1+G \end{bmatrix}\begin{bmatrix} R_\theta \\ R_\phi \end{bmatrix} \cdot \frac{G}{(1+G)^2 - J^2}. \tag{13}$$

Should $R_\theta = R_\phi$, (11) and (12) reduce to

$$\frac{C_\theta}{R_\theta} = \frac{G}{1+G-J} \tag{14}$$

$$\frac{C_\phi}{R_\phi} = \frac{G}{1+G-J}. \tag{15}$$

Or, in matrix form,

$$\begin{bmatrix} C_\theta \\ C_\phi \end{bmatrix} = \begin{bmatrix} R_\theta \\ R_\phi \end{bmatrix} \cdot \frac{G}{1+G-J}. \tag{16}$$

Should $R_\theta = -R_\phi$, (11) and (12) reduce to

$$\frac{C_\theta}{R_\theta} = \frac{G}{1+G+J} \tag{17}$$

$$\frac{C_\phi}{R_\phi} = \frac{G}{1+G+J}. \tag{18}$$

Or, in matrix form,

$$\begin{bmatrix} C_\theta \\ C_\phi \end{bmatrix} = \begin{bmatrix} R_\theta \\ R_\phi \end{bmatrix} \cdot \frac{G}{1+G+J}. \tag{19}$$

The latter case is equivalent to the former case with J negative. The analysis of the system performance thus becomes an analysis of the general case transfer function which contains $(1+G)^2 - J^2 = (1+G-J)(1+G+J)$ in the denominator, and the special case transfer function which contains $1+G-J$ in the denominator, where J may be either positive or negative.

Note that the location of the cross-coupling in the circuits is immaterial. That is, the product, G_1G_2, always occurs and is equal to G, the forward transfer function, without cross-coupling. Inasmuch as G_1 and G_2 do not appear independently, but only as products reducible to G, it will not be necessary to determine G_1 and G_2 individually.

Note further that in transfer functions of this nature, the reference term, R_θ or R_ϕ, is basically a normalizing term for the plane of action of the equation. Thus, any cross-coupling term generating from another input must occur in a normalized form (*i.e.*, R_ϕ effects in the R_θ-plane appear as R_ϕ/R_θ).

In spite of the generalized nature of the algebraic model thus far developed, there are some inherent limitations that bear mentioning. First, the model requires that each system component be describable in performance by linear differential equations in real time. Second, the system has been assumed to be identical in both operating planes, and the cross-coupling transfer function J has been assumed to be identical also. These assumptions have been made to simplify the treatment and the examples, and are considered justifiable on the basis that the majority of two-dimensional systems in existence are designed to be identical in both planes of action. Further, it is believed that the treatment provided in this paper can provide guide lines and techniques to be followed in the analysis of non-symmetrical two-dimensional systems.

III. Stability

In the analysis of cross-coupling effects on the performance of two-dimensional, orthogonal, feedback control systems consideration must be given to the stability of the total system. This must be done in order to establish whether the system is or is not stable, whether or not gain and phase margins exist, and to provide a basis for manipulating the response of the system so as to secure acceptable over-all response characteristics in

the presence of cross-coupling which cannot be eliminated. It is the purpose of this section to present an exposition of how such an analysis may be made when the cross-coupling transfer function may be expressed by a linear, differential equation.

Returning to the algebraic model developed in Section II, consider that it is a representation of an automatically tracking radar system in which the component and cross-coupling transfer functions may be frequency-sensitive. The general expression (13) may be investigated for system stability.

$$\begin{bmatrix} C_\theta \\ C_\phi \end{bmatrix} = \begin{bmatrix} 1+G & J \\ J & 1+G \end{bmatrix} \begin{bmatrix} R_\theta \\ R_\phi \end{bmatrix} \cdot \frac{G}{(1+G)^2 - J^2} \cdot \tag{13}$$

To determine the stability of the general system represented by (13), it is necessary first to establish that the zeros of the denominator are the zeros of $(1+G)^2-J^2$ (*i.e.*, no poles are contributed by the numerator). This may be done by letting

$$G = \frac{n(G)}{d(G)} \quad \text{and} \quad J = \frac{n(J)}{d(J)} \cdot$$

Then

$$C_\theta = \frac{R_\theta \dfrac{n(G)}{d(G)}\left[1 + \dfrac{n(G)}{d(G)}\right] + R_\phi \dfrac{n(G)n(J)}{d(G)d(J)}}{\left[1 + \dfrac{n(G)}{d(G)}\right]^2 - \left[\dfrac{n(J)}{d(J)}\right]^2} \cdot \tag{20}$$

This expression may be resolved to

$$C_\theta = \frac{R_\theta n(G)[d(G)+n(G)][d(J)]^2 + R_\phi n(G)n(J)d(G)d(J)}{[d(G)+n(G)]^2[d(J)]^2 - [n(J)]^2[d(G)]^2} \cdot \tag{21}$$

The zeros of the denominator of this expression are thus the zeros of the denominator of the general cross-coupled system transfer function (13), and the stability of the general system may be analyzed by studying the zeros of $(1+G)^2-J^2$.

It is necessary next, following Nyquist's stability criterion, to determine that none of the zeros of $(1+G)^2-J^2$ lie in the positive half of the complex s-plane (condition for stability). Although this may be interpreted as stating that an $s=j\omega$ (Nyquist) plot of $2G+G^2-J^2$ must follow the usual Nyquist rules regarding encirclement of the -1 point, the application of graphical techniques to determine a plot of $2G+G^2-J^2$ does not appear to be nearly so convenient as it is to factor the expression $(1+G)^2-J^2$ into $(1+G-J)(1+G+J)$. For the factored form of the expression to be equal to zero, one of the two factors must be equal to zero. These two possibilities are the same condition considered in Section II as special cases of the general system [(16) and (19)]. Therefore, stability conditions may be stipulated for these special cases, and these conditions should suffice to determine the stability of the general system. That is, the system will be stable under all conditions if it is analyzed as a special case of the general system, and is determined to be stable for both positive and negative cross-coupling ($\pm J$).

In order to determine the stability of such a system it is necessary, following Nyquist's stability criterion, to determine that none of the zeros of the denominator, $1+G(s)+J(s)$, lie in the positive half of the complex s-plane (condition for stability), *i.e.*, when

$$1 + G(s) + J(s) = \frac{(s-r_1)(s-r_2)\cdots(s-r_n)}{(s-r_a)(s-r_b)\cdots(s-r_m)} = 0, \tag{22}$$

it is necessary for stability that none of the roots, $r_1, r_2, r_3, \cdots, r_n$, have positive real parts. There is no such restriction on the other roots, $r_a, r_b, r_c, \cdots, r_m$. The stability criterion may be interpreted for this special case system as requiring that all -1 values of $G(s)+J(s)$ be located in the left (or negative real) half of the s-plane.

If a complex-plane curve for $G(s)+J(s)$ can be constructed with $s=j\omega$ for all values of ω from $-\infty$ to $+\infty$, then the Nyquist stability criterion may be applied. A vector between the $(-1+j0)$ point to the $(G(s)+J(s))$ plot may be observed to determine the net number of counterclockwise revolutions of the vector as ω varies from $-\infty$ to $+\infty$, and then determining whether this net number of counterclockwise revolutions is equal to the number of poles of the open loop transfer function in the right half plane (for stability) or not (instability). Note that this is an abbreviated version of the Nyquist stability criteria which includes the assumption that the net number of counterclockwise revolutions will be determined as stated in the procedure outlined on p. 145 of [7]. A zero of $1+G(s)+J(s)$ with positive real parts will produce a clockwise revolution of the vector, and a pole of $1+G(s)+J(s)$ with positive real parts will produce a counterclockwise revolution of the vector. It is usually assumed that $G(s)$ and $J(s)$ are by themselves stable, *i.e.*, $r_a, r_b, \cdots, r_m$, the poles of $1+G(s)+J(s)$ all lie in the negative half plane. For systems in which $1+G(s)+J(s)$ contains poles in the positive half plane, it is necessary for stability that the number of counterclockwise revolutions of the Nyquist vector equal the number of poles in the positive half plane. Note that the following two limitations are implicit in using the Nyquist stability criteria; 1) the system must be capable of being represented by a system of linear differential equations with constant coefficients, and 2) the limit of $G(s)$, and of $J(s)$, must approach a constant or zero as s approaches ∞. The latter limitation is inherently present physically in linear systems. Mathematically, it is required that for any physical system the transfer function contain a higher power polynomial in its denominator than in its numerator.

In order to make a complex plane plot of the function $G(s)+J(s)$ with $s=j\omega$, and allowing ω to go from $-\infty$

to $+\infty$, the following procedure appears to be the least tedious available with presently known mathematical techniques:

1) Make an s-plane plot of the poles and zeros for $G(s)$. By graphical construction, take values of $s=j\omega$ as ω goes from $-\infty$ to $+\infty$, determine the magnitude and phase angle for $G(j\omega)$, and plot these values on a complex plane. This will produce the typical Nyquist diagram, which is ordinarily analyzed for stability by determining enclosures of the -1 point, or the number of rotations of the vector between $(-1+j0)$ and the $G(j\omega)$ curve as ω goes from $-\infty$ to $+\infty$.

2) Repeat the foregoing procedure to obtain a similar plot for $J(j\omega)$ on the same complex plane.

3) Indicate identical values of ω at various points on both the $G(j\omega)$ and $J(j\omega)$ plots, thus producing a plot such as Fig. 1.

4) Sum the two curves vectorially for several values of ω, and make a new complex plane plot which will be a plot of $G(j\omega)+J(j\omega)$ as ω goes from $-\infty$ to $+\infty$. The plot may have the form of Fig. 2.

This plot may then be analyzed relative to the $-1+j0$ point following the usual Nyquist method.

The example used depicted a $J(s)$ of the nature

$$J(s) = \frac{J_K}{(\tau_a s + 1)(\tau_b s + 1)}, \tag{23}$$

and a $G(s)$ of the nature

$$G(s) = \frac{G_K}{s(\tau_1 s + 1)(\tau_2 s + 1)}. \tag{24}$$

This combination does not appear to give rise to any serious stability problems. However, had $J(s)$ been of the nature

$$J(s) = \frac{J_K}{s(\tau_a s + 1)(\tau_b s + 1)\epsilon^{\tau_c s}} \tag{25}$$

or

$$J(s) = \frac{J_K(\tau_d s + 1)(\tau_e s + 1)(\tau_f s + 1)}{(\tau_a s + 1)(\tau_b s + 1)(\tau_c s + 1)}, \tag{26}$$

with

$$\tau_a, \tau_b, \tau_c \gg \tau_d, \tau_e, \tau_f,$$

a serious stability problem might occur.

Consider the individual plots of step 3) in the procedure shown in Fig. 3 (next page).

A summation of these two plots may yield a plot such as Fig. 4. Fig. 4 indicates how the mere addition of the cross-coupling transfer function may easily lead to an absolute instability which may or may not be eliminated by a simple gain decrease in $G(s)$, the stability being dependent also on the gain and phase characteristics of $J(s)$. A similar summation should be made for the other special case obtained by taking J as negative in sign. Such a plot is depicted in Fig. 5.

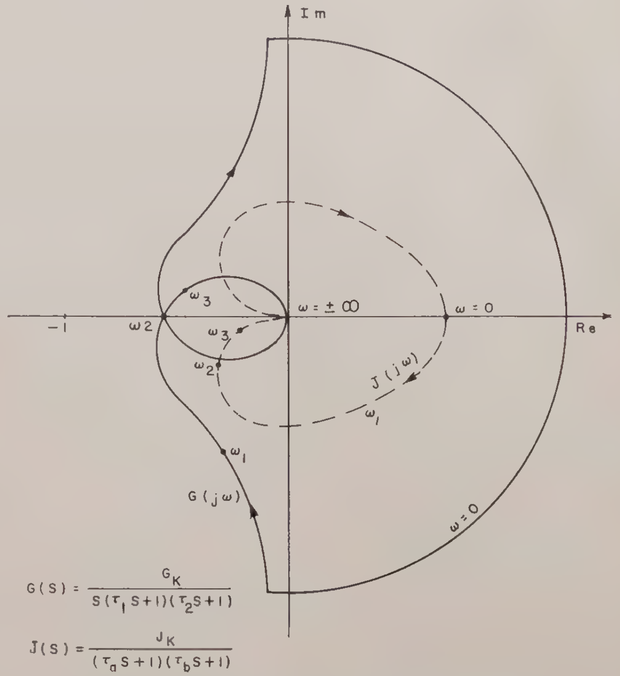


Fig. 1—Complex-plane plots of $G(j\omega)$ and $J(j\omega)$.

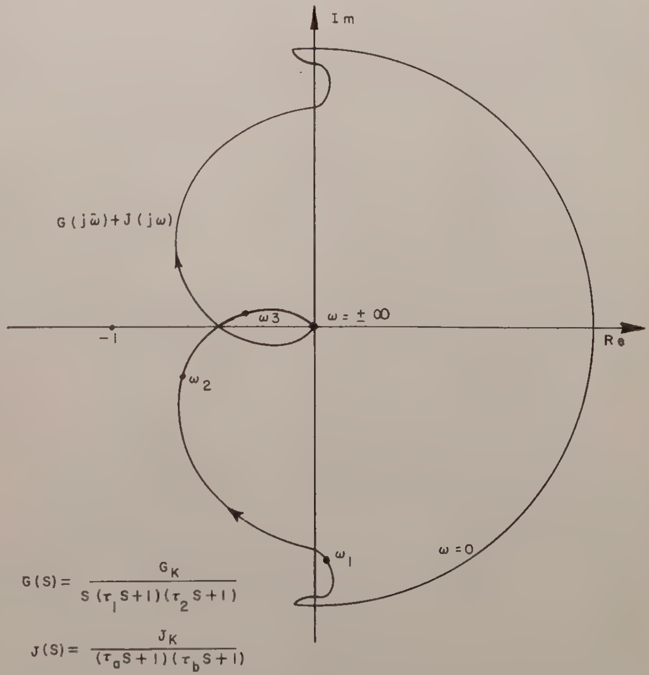


Fig. 2—Complex-plane plot of $G(j\omega)+J(j\omega)$; system not made unstable by J addition.

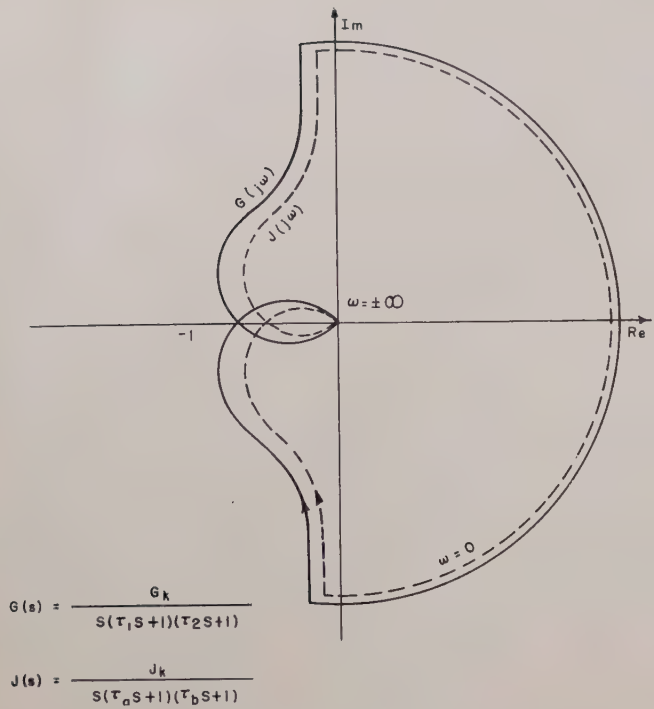
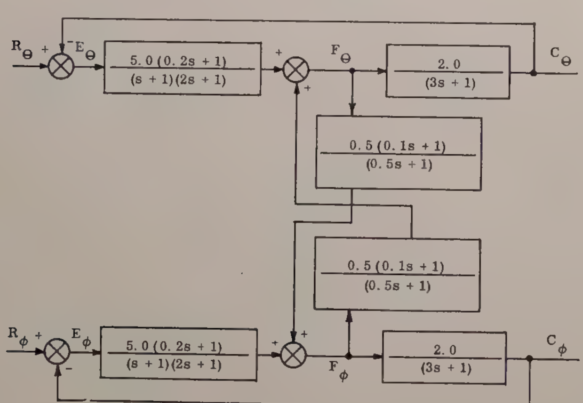


Fig. 3—Complex-plane plots of $G(j\omega)$ and $J(j\omega)$.

It should be remembered that both special cases ($\pm J$) must be stable for the system to be stable, and that the use of cross-coupling to make one case more stable will generally tend to make the companion case less stable. The stability margin of the total system will usually not be greater than either special case ($\pm J$) stability margin.

To illustrate the analytical methods of this section further, consider a positional system,



$$G(s) = \frac{5.0(0.2s + 1)}{(s + 1)(2s + 1)} \cdot \frac{2.0}{3s + 1}$$

$$= \frac{10.0(0.2s + 1)}{(s + 1)(2s + 1)(3s + 1)}, \tag{27}$$

$$J(s) = \frac{0.5(0.1s + 1)}{(0.5s + 1)} \cdot \tag{28}$$

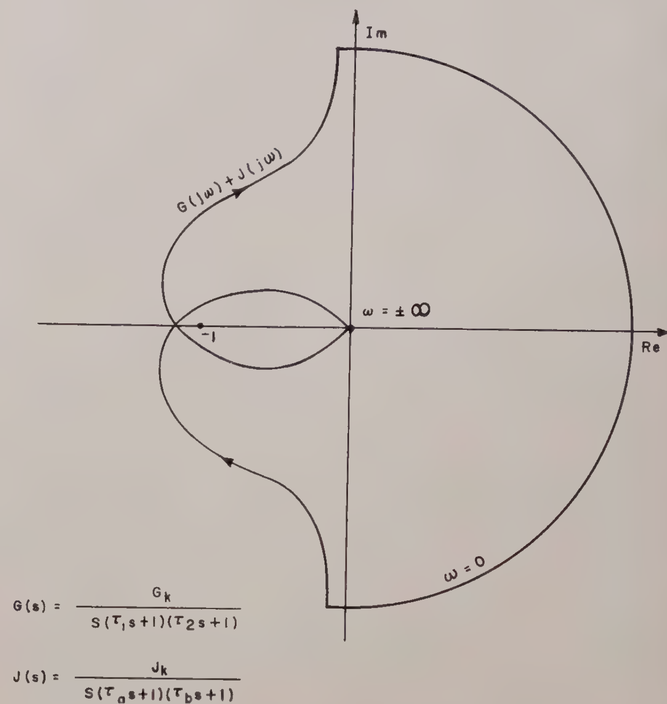


Fig. 4—Complex-plane plot of $G(j\omega)+J(j\omega)$: system made unstable by J addition.

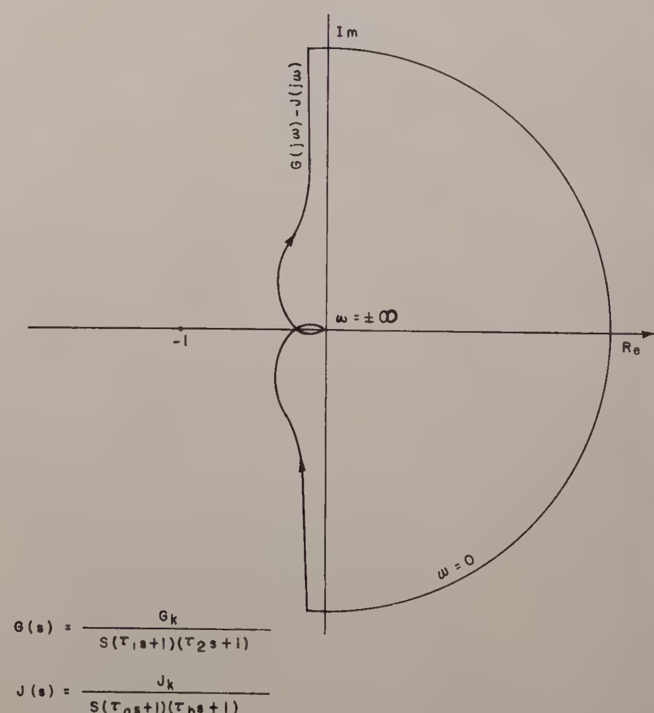


Fig. 5—Complex-plane plot of $G(j\omega)-J(j\omega)$; system not made unstable by J subtraction.

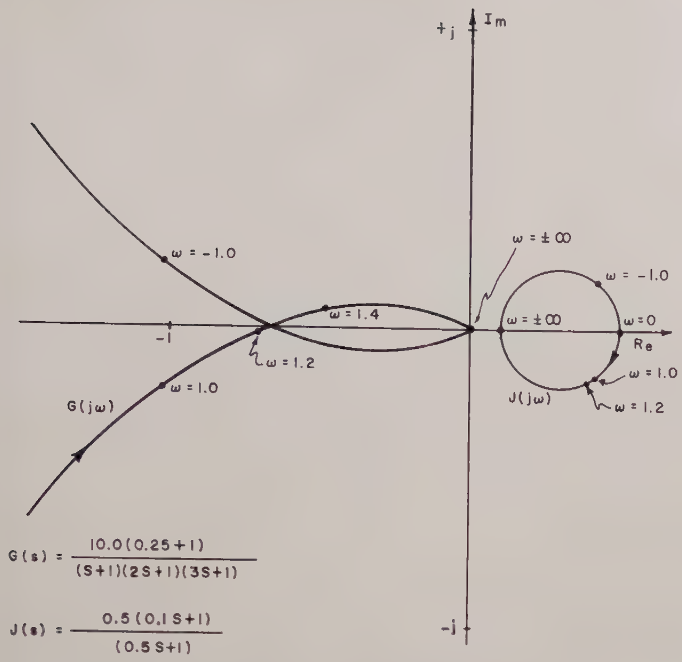


Fig. 6—Complex-plane plot of $G(j\omega)$ and $J(j\omega)$ for time constant example.

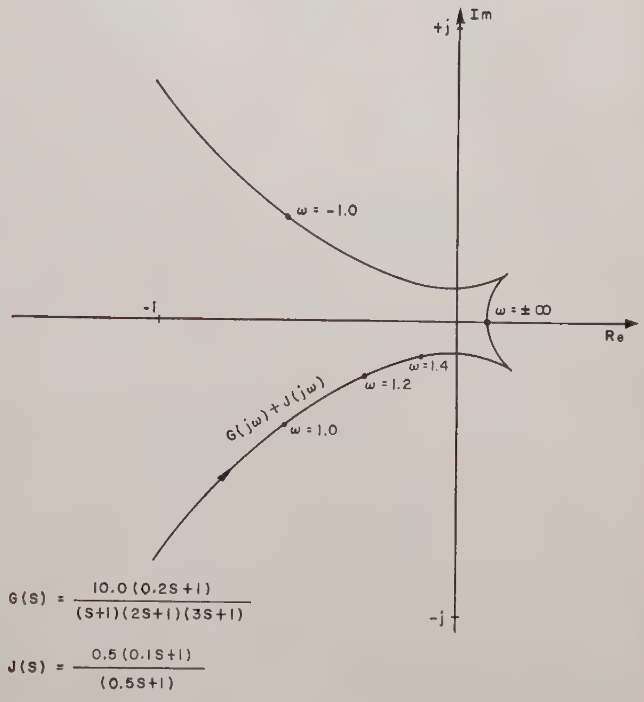


Fig. 7—Complex-plane plot of $G(j\omega)+J(j\omega)$ for time constant example.

$G(j\omega)$ and $J(j\omega)$ may be plotted as illustrated in Fig. 6. Then, from these plots we may plot $G(j\omega)+J(j\omega)$, Fig. 7, and $G(j\omega)-J(j\omega)$, Fig. 8. These plots illustrate again, using actual time constant values, the manner in which cross-coupling may cause a stable system to become unstable.

A demonstration of how cross-coupling may be of a beneficial nature in a system, and therefore purposely introduced, may be obtained by considering a system whose open loop, one axis transfer function may be

$$G(s) = \frac{G_K(\tau_3 s+1)(\tau_4 s+1)}{s(\tau_1 s+1)(\tau_2 s+1)(\tau_5 s+1)} \tag{29}$$

with

$$\tau_1 > \tau_2 > \tau_3 > \tau_4 > \tau_5,$$

and such that a complex plane plot will appear as depicted in Fig. 9 (next page). Such a system may be immediately judged as unstable. However, consider introtroducing cross-coupling of the nature,

$$J(s) = \frac{J_K(\tau_d s+1)(\tau_e s+1)(\tau_f s+1)}{(\tau_a s+1)(\tau_b s+1)(\tau_c s+1)} \tag{30}$$

with

$$\tau_a,\ \tau_b,\ \tau_c > \tau_d,\ \tau_e,\ \tau_f$$

and $J_K > 1$, and such that a complex plane plot will appear as depicted in Fig. 9. Making the graphical summation of $G(j\omega) \pm J(j\omega)$ should then yield the complex

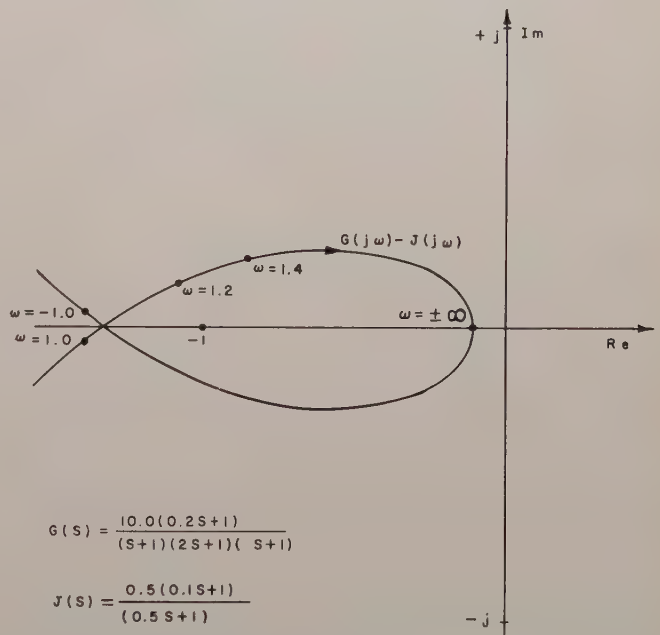


Fig. 8—Complex-plane plot of $G(j\omega)-J(j\omega)$ for time constant example.

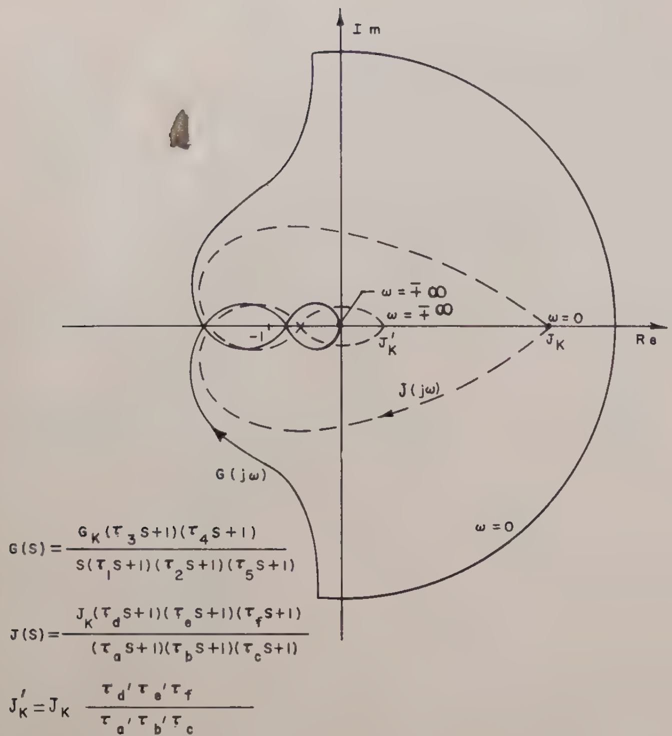


Fig. 9—Complex-plane plot of unstable $G(j\omega)$ and $J(j\omega)$ introduced to stabilize system.

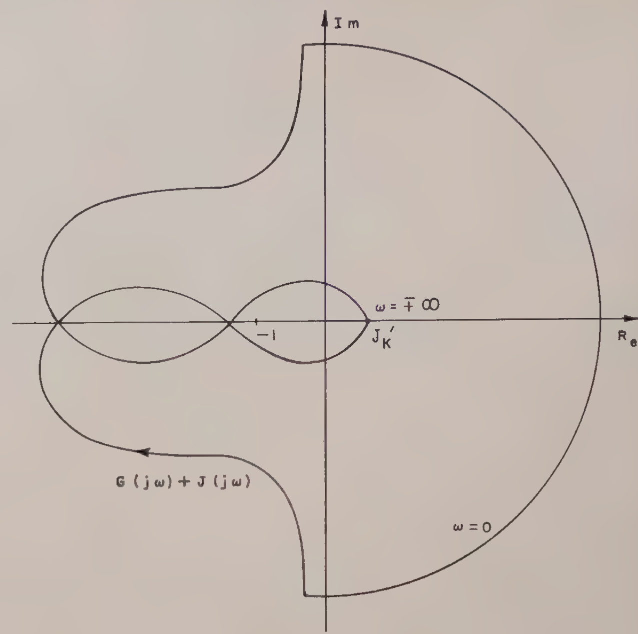


Fig. 10—Complex-plane plot of $G(j\omega)+J(j\omega)$ for stabilized system of Fig. 9.

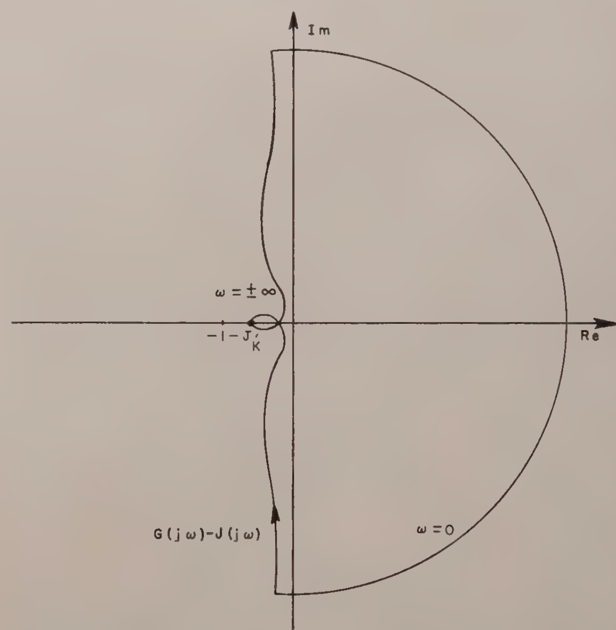


Fig. 11—Complex-plane plot of $G(j\omega)-J(j\omega)$ for stabilized system of Fig. 9.

plane plots depicted in Figs. 10 and 11; and by comparison with Fig. 9 it may be observed that the unstable system has been made stable by the introduction of cross-coupling. Thus, cross-coupling may be usefully introduced for the purpose of making an unstable system stable (although consideration must be given to the seriousness of adverse effects such as error vector distortion).

IV. Conclusions

The stability of cross-coupled systems may be investigated by graphical techniques following the procedure outlined in Section III. The total system stability is dependent on the forward transfer function and the cross-coupling transfer function. The presence of cross-coupling in an otherwise stable two-dimensional system may cause the system to become unstable. Unstable two-dimensional systems may be made stable by the introduction of suitable cross-coupling transfer functions to the system, as illustrated in Section III.

The work of other authors, particularly [1]–[6], has been added to by this paper in that a general treatment is now provided for the analysis of the stability of any cross-coupled system in which the cross-coupling transfer function may be expressed by a linear differential equation, and the analysis may be accomplished by complex plane methods.

Bibliography

[1] M. Golomb and E. Usdin, "A theory of multidimensional servo systems," *J. Franklin Inst.*, vol. 253, pp. 29–57; January, 1952.

[2] R. H. Merson, "A Generalization of the Nyquist Stability Criterion with Particular Reference to Phasing Error," Royal Aircraft Establishment, England, Tech. Note No. GW 316 CP No. 330; 1954.

[3] A. A. Krasovsky, "On two-channel control systems described by equations with complex parameters," *Avtomat. i Telemekh.*, vol. 18, pp. 126–136; February, 1957.

[4] R. J. Kavanagh, "Multivariable control system synthesis," *Trans. AIEE*, pt. 2, vol. 77, pp. 425–429; November, 1958.

[5] H. Freeman, "Stability and physical realizability considerations in the synthesis of multipole control systems," *Trans. AIEE*, pt. 2, vol. 77, pp. 1–5; March, 1958.

[6] D. J. Povejsil and A. M. Fuchs, "A method for the preliminary synthesis of complex multiloop control system," *Trans. AIEE*, pt. 2, vol. 74, pp. 129–134; July, 1955.

[7] H. Chestnut and R. W. Mayer, "Servomechanisms and Regulating System Design," John Wiley and Sons, Inc., New York, N. Y., vol. 1; 1951.

Stabilization of Linear Multivariable Feedback Control Systems*

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Summary—**Stabilization techniques for single variable feedback control systems are well known. For multivariable feedback control systems, comparable methods are still in the initial stage of development. A method of system stabilization is discussed, which reduces the problem to essentially that of a single variable system. This method uses a compensating matrix which is either the inverse system- or transposed system-matrix. An example of this method of system stabilization is given and its usefulness discussed.**

Introduction

STABILIZATION techniques for single variable feedback control or servo—systems have been brought to a high degree of development. For multivariable servo systems, the development of comparable techniques is still in its initial stages. It is of interest to give a brief discussion of the relevant material available in the literature. Consider the linear multivariable servo system shown in Fig. 1. X, Y and V are single column matrices of n elements each where X is the input, Y is the controlled output, and V is the external disturbance. A, C, G and H are square matrices of n^2 elements each. A represents the system; C is the compensation matrix; G represents the high gain power amplifier stages; and H is a feedback matrix. The elements of these matrices are the Laplace transforms of the respective time functions. These matrices are related by the following equations:

$$U = C(X - Z), \tag{1}$$

$$W = UG + V, \tag{2}$$

$$Y = AW, \tag{3}$$

$$Z = HY. \tag{4}$$

By a process of elimination it is easy to obtain

$$(I + AGCH)Y = (AGC)X + AV, \tag{5}$$

where I is the unit matrix.

A system of this type, where G and H are diagonal, has been extensively discussed in the literature as an example of noninteracting controls.[1–8] The condition for noninteraction is $X = Z$. From the servo system point of view, this is achieved by having a high loop gain. The system is then relatively insensitive to both the external disturbance matrix V and to the interaction of signals. In this sense, every multivariable servo system is noninteracting. However, this is not the approach taken in the references cited. To understand this situation, consider the problem of determining a compensating matrix C in order to obtain a stable system with a high gain in the elements of G. One possibility is to choose

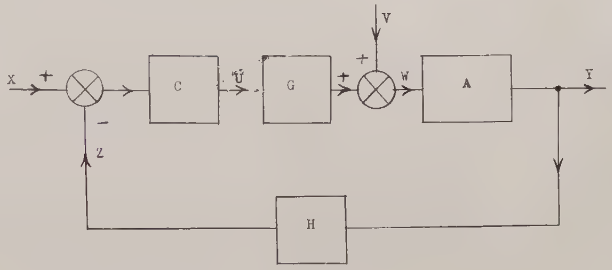

Fig. 1—Multivariable servo system.

$$C = DA^{-1}, \tag{6}$$

where D is diagonal. For this choice of C we have

$$AGCH = DGH, \tag{7}$$

which is diagonal. Eq. (5) can now be written as

$$Y_k = \frac{D_{kk}G_{kk}}{1 + D_{kk}G_{kk}H_{kk}} \cdot X_k + \frac{(AV)_k}{1 + D_{kk}G_{kk}H_{kk}}; \quad k = 1, \cdots, n. \tag{8}$$

This can be simplified to

$$Y_k = R_{kk}X_k - (R_{kk}H_{kk} - 1)(AV)_k \tag{9}$$

where

$$R_{kk} = \frac{D_{kk}G_{kk}}{1 + D_{kk}G_{kk}H_{kk}}. \tag{10}$$

This particular choice of C diagonalizes the open loop transfer matrix $AGCH$. Eq. (9) can now be treated as n independent servo systems, and conventional single variable techniques can be used to determine suitable compensating elements for D_{kk} or G_{kk}. These single variable techniques are the extensively developed methods based on the root locus, Nyquist plots, and phase and gain margins.

The result (9) is given in the cited references as the condition for noninteraction. This terminology is con-

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[1] H. Tsien, "Engineering Cybernetics," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 53–69; 1954.

[2] R. Kavanagh, "Noninteracting controls in linear multivariable systems," *Trans. AIEE*, vol. 76, pt. 2, pp. 95–100; May, 1957.

[3] A. Boksenbom and R. Hood, "General Algebraic Method Applied to Control Analysis of Complex Engine Types," Natl. Advisory Committee for Aeronautics, Washington, D. C., Rept. No. 980; April, 1949.

fusing since every high gain multivariable servo system is noninteracting. H. Freeman, in the discussion of Kavanagh's paper,[2] has criticized the interpretation of noninteraction from a different point of view.

In a separate paper, Freeman[4] discusses the synthesis problem and introduces the concept of "independent output restoration." This terminology is unnecessary since the open loop transfer matrix is diagonalized exactly as is done in the system just discussed. The difference is simply that, in Freeman's case, H is not diagonal. The synthesis and related problems are also discussed by Freeman[5] and Kavanagh.[6] Both these latter papers rely on having determined, by some means, the desired closed loop transfer matrix K. A compensation matrix C is then determined to realize this K. This approach has been criticized by F. Nesline in the discussion of Kavanagh's paper.[6] Before a suitable choice of K can be made, the dynamics of the closed loop system must be identified and approximated. It is necessary to investigate the response of the fixed elements of the system with respect to some criterion. A simple criterion is that of having a high loop gain consistent with a stable dynamic response. The question of stability and response of multivariable servo systems has received little attention. Very special cases are discussed by Golomb and Usdin[7] and Pelegrin.[8]

Multivariable Servo Systems

To avoid undue complexity in discussing the stability problem, we will set $V=0$ and $H=I$. The system matrix is

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}, \tag{11}$$

where the A_{jk}'s are considered to be fixed transfer functions. G will be taken to be diagonal. The simplifying assumption is made that all power amplifier stages are identical and have a transfer function $G(s)$. This involves no restrictions since, through suitable compensation networks, this can always be achieved. Hence

$$G = G(s) \cdot I. \tag{12}$$

The uncompensated multivariable servo system is shown in Fig. 2. The matrix equations for this system are

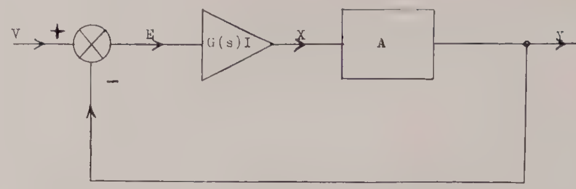

Fig. 2—Uncompensated multivariable servo system.

$$Y = AX, \tag{13}$$

$$E = V - Y = V - AX, \tag{14}$$

$$X = G(s)E = -\frac{E}{\lambda}, \tag{15}$$

where we have set

$$\lambda = -\frac{1}{G(s)} \cdot \tag{16}$$

From (13)–(16), it is easy to obtain

$$(A - \lambda I)X = V. \tag{17}$$

Eq. (17) will be used in the discussion of system stability.

System Stability

The stability of the system can be determined by means of the Nyquist criterion. Let

$$\lambda_k(s); \qquad k = 1, \cdots, n$$

be the eigenvalues of the A-matrix, *i.e.*, the values of λ which satisfy the equation

$$|A - \lambda \cdot I| = 0. \tag{18}$$

The system is stable if there are no values of s in the right half s-plane which are roots of (18), since these are the poles of the closed loop system. To determine whether this is the case, consider the function $R_k(s)$ defined by

$$R_k(s) = \frac{1}{\lambda_k(s)} + G(s). \tag{19}$$

From (16), (18), and (19) it is easily seen that the roots of (18) are the zeros of

$$R_k(s) = 0. \tag{20}$$

The condition for stability is that the zeros of (20) must all lie in the left half s-plane. This can be determined by considering the Nyquist plot of $R_k(j\omega)$ and application of

$$N = Z - P, \tag{21}$$

[4] H. Freeman, "A synthesis method of multipole control systems," *Trans. AIEE*, vol. 76, pt. 2, pp. 28–31; March, 1957.
[5] H. Freeman, "Stability and physical realizability considerations in the synthesis of multipole control systems," *Trans. AIEE*, vol. 77, pt. 2, pp. 1–5; March, 1958.
[6] R. Kavanagh, "Multivariable control system synthesis," *Trans. AIEE*, vol. 77, pt. 2, pp. 425–429; November, 1958.
[7] M. Golomb and E. Usdin, "A theory of multidimensional servo systems," *J. Franklin Inst.*, vol. 253, pp. 28–57; January, 1952.
[8] J. Gill, M. Pelegrin, and P. Decaulne, "Feedback Control Systems," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 370–373; 1959.

where P is the number of poles and Z the number of zeros of $R_k(s)$ in the right half s-plane respectively. N is the number of positive revolutions of the radius vector $R_k(j\omega)$. If $Z=0$, the system is stable; if $Z>0$, there are values of s in the right half s-plane which are roots of (18), and the system is unstable.

System Stabilization

A general theory of the stabilization of multivariable servo systems would be very complicated, and its usefulness for design purposes limited, due to the computational difficulties involved. A method of multivariable servo system stabilization is discussed which reduces the problem to essentially that of a single variable system. It is shown how this objective can be accomplished byasuitable choiceof thecompensating matrix C.

The system already discussed with reference to Fig. 1 and (8) indicates one possibility. Eq. (6) shows that the compensating matrix is essentially the inverse of the system matrix. This coice of C leads to n independent systems. The stability and response of each of these can be investigated by conventional single variable techniques and suitable compensating elements determined. This method appears to be very general. Some of its possible disadvantages in a practical system will be discussed later. The choice $C=A^{-1}$ for the compensation matrix will be called type I compensation.

Consider now the problem of stabilizing the system discussed with reference to Fig. 2 and (16) and (17). The system eigenvalues $\lambda_k(s)$ may lie anywhere in the s-plane. To ensure system stability and reduce the complexity of the stabilization problem, the C-matrix will be chosen so as to localize the system eigenvalues to specified regions of the s-plane. To see how this can be accomplished consider the augmented system

$$(B - \lambda I)\cdot\begin{pmatrix} X \\ W \end{pmatrix} = \begin{pmatrix} O \\ V \end{pmatrix}, \tag{22}$$

where

$$B = \begin{pmatrix} I & -C \\ A & O \end{pmatrix}. \tag{23}$$

This is equivalent to the two matrix equations,

$$CW = (1-\lambda)X, \tag{24}$$

$$AX = \lambda W + V. \tag{25}$$

There are two possible interpretations of this augmented system. One is similar to Fig. 2, with V, X and A being replaced by

$$\begin{pmatrix} O \\ V \end{pmatrix}, \quad \begin{pmatrix} X \\ W \end{pmatrix}$$

and B, respectively. This representation is based on (22). Eqs. (24) and (25) lead to the representation in Fig. 3.

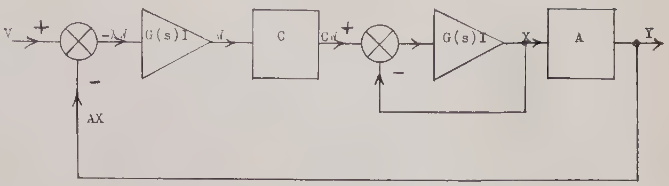

Fig. 3—Augmented multivariable servo system.

[It is important to note that λ is defined by (16).] The use of an augmented matrix of this type was introduced by Goldberg and Brown[9] to stabilize the solution of linear equations on an analog computer. In this application, the system matrix A contained only real numbers, and C was chosen to equal A^T, the transpose of A.

Eqs. (24) and (25) can be simplified by eliminating W. This leads to

$$(CA - \gamma I)X = CV, \tag{26}$$

where

$$\gamma = \lambda(1-\lambda). \tag{27}$$

Consider now the eigenvalues $\gamma_k(k=1, \cdots, n)$ of the matrix CA. If λ_k is an eigenvalue of the augmented matrix B, (27) gives

$$\gamma_k = \lambda_k(1-\lambda_k). \tag{28}$$

This can be solved for λ_k to give

$$\lambda_k = \tfrac{1}{2}(1 \pm j\sqrt{4\gamma_k - 1}). \tag{29}$$

To see how C can restrict the location of the eigenvalues of B in the s-plane, suppose that CA is a positive hermitian matrix. Then γ_k is real and positive (see Appendix III). The eigenvalue locus for all augmented systems is then located on the two straight lines.

$$\lambda_k = \begin{cases} \tfrac{1}{2}(1 \pm \sqrt{1-4\gamma_k}) & 0 \leqq \gamma_k \leqq \tfrac{1}{4} \\ \tfrac{1}{2}(1 \pm j\sqrt{4\gamma_k - 1}) & \gamma_k \geqq \tfrac{1}{4}. \end{cases} \tag{30}$$

These two lines will be called the real locus.

Since the elements of the C-matrix must consist of physically realizable transfer functions, the question immediately arises as to how this is to be achieved in addition to ensuring that CA be hermitian. A simple possible choice for C is

$$C = hA^{-1}, \tag{31}$$

where h is a real constant >0. This is type I compensation. In this case, $CA=hI$. Hence $\gamma_k = h$ and

$$\lambda_k = \tfrac{1}{2}(1 \pm \sqrt{1-4h}). \tag{32}$$

[9] E. Goldberg and G. Brown, "An electronic simultaneous equation solver," *J. Appl. Phys.*, vol. 19, pp. 338–345; April, 1948.

Let λ_1 and λ_2 be the two values of the right hand side of (32). The two points, $-1/\lambda_1$ and $-1/\lambda_2$ in the s-plane will be called critical points corresponding to the point -1 in the one variable case. The conventional single variable Nyquist criterion is now applicable. This means that, provided $P=0$ [see (21)], the encirclement of the critical points by the locus $G(j\omega)$ is to be avoided by a suitable gain and phase margin.

A second possible choice is $C=A^*$, where A^* is the adjoint of A (see Appendix III). However, the elements of A^* are not generally physically realizable. The conditions can be partially met by simply choosing $C=A^T$. For s real $A^*=A^T$ and the $\lambda_k(s)$ locus will lie on the real locus. For $s=j\omega$, the Nyquist locus $\lambda_k(j\omega)$ will be symmetrical about the real locus as shown in Fig. 4 (for simplicity only half of the locus is drawn). Hence, this choice of C also restricts the eigenvalue locus. This type of compensation will be called type II.

The stability can once more be determined by application of (21). For example, suppose $P=0$ and the Nyquist plots of $G(j\omega)$ and $-1/\lambda_k(j\omega)$ are as shown in Fig. 4. A locus must be plotted for each $k=1, \cdots, n$. $Z=N$ is determined in each case from the number of positive revolutions of the radius vector $R_k(j\omega)$. A simple approximation can be obtained by considering the largest value λ_{max} of all the λ_k's on the real locus. $-1/\lambda_{max}$ can then be taken as the critical point and single variable techniques may be usefully applied.

Physical Realizability

With type I compensation, C is physically realizable if the elements of A^{-1} are physically realizable as transfer functions. This is the case if A^{-1} has no poles (and hence if A has no zeros) in the right half s-plane. The C-matrix with type II compensation is always physically realizable.

Conclusion

By suitable choice of the transfer function of the power amplifier $G(S)$, type I compensation will evidently always lead to a stable system, provided A has no zeros in the right half s-plane. The dynamic response may be somewhat sluggish due to the multiplicity of equal roots. This can be seen from (32). The augmented system has n eigenvalues each equal to λ_1, and n eigenvalues each equal to λ_2, respectively. The zeros of (20), which are the poles of the closed loop system, will have a corresponding multiplicity. A practical disadvantage of this method is that it evidently depends on having the actual system closely approximated by a linear one over the entire range of operation.

Type II compensation may not always lead to a stable system. However, it has the practical advantage in that the C-matrix may consist of physical analogs or models of the system. The method can then be applied to nonlinear multivariable servo systems which can be linearized by describing function methods.

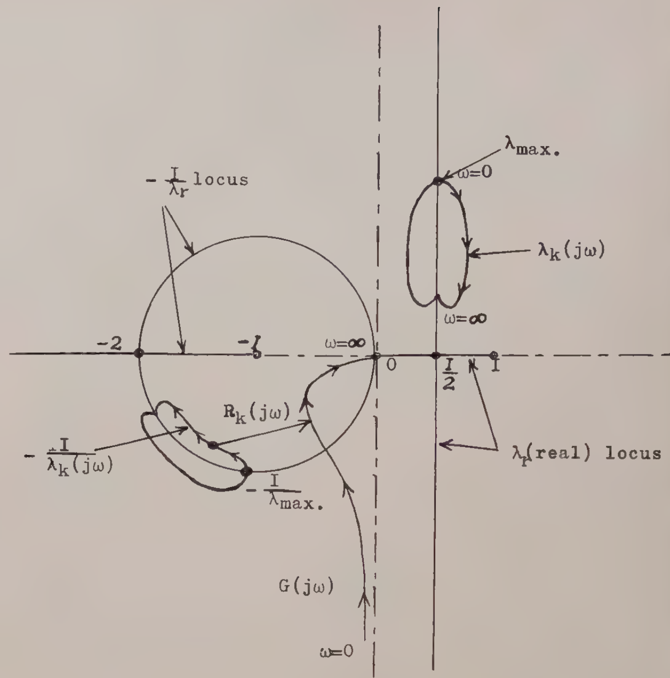

Fig. 4—λ-plane.

Appendix I

As an example, consider a system whose matrix is

$$A = \begin{pmatrix} Y & -1 \\ 1 & Y \end{pmatrix}, \tag{33}$$

where

$$Y = \frac{1}{1+ST}. \tag{34}$$

For the power amplifier, assume

$$G(s) = \frac{Kv}{s(1+s\tau)}. \tag{35}$$

The performance of this multivariable system will be judged by comparing it with a reference system. This system is chosen to be a conventional single variable servo system having the same power amplifier. For the reference system, the open loop transfer function is then given by (35). If a relative damping ratio of $\zeta=0.6$ is chosen for the poles of the closed loop reference system, it is found that

$$K_v\tau = 0.945. \tag{36}$$

The velocity error for a unit ramp input is

$$\epsilon_1(\infty) = 1.06\tau. \tag{37}$$

To determine the stability of the unmodified multivariable servo system, the system eigenvalues, λ_1 and λ_2, must first be found from (18). These are

$$\lambda_1 = Y + j$$

$$\lambda_2 = Y - j. \tag{38}$$

The poles of the closed loop system are the roots of

$$\lambda_1 = -\frac{1}{G(s)} = -\frac{s(1+s\tau)}{K_v} = \frac{1}{1+sT} + j, \quad (39)$$

and a similar equation with λ_2. By substituting $sT=u$, $\tau/T=a$, and $K_vT=b$, the following cubic equation is obtained:

$$u(1+au)(1+u) + jb(1+u) + b = 0. \quad (40)$$

Assume now that $a \ll 1$. One root of (40) is then approximately $-1/a$, and the other two roots are solutions of the quadratic

$$u^2 + (1+jb)u + (1+j)b = 0. \quad (41)$$

The gain K_v is to be chosen so that the relative damping ratio of the roots of the system will be ≥ 0.6. Solving (41) under this constraint leads to the following results:

$$\begin{aligned} b &= 0.29 \\ u_1 &= -0.28 - j0.47, \\ u_2 &= -0.72 + j0.18, \\ u_3 &\cong -\frac{1}{a}. \end{aligned} \quad (42)$$

The eigenvalue λ_2 will evidently result in the conjugate complex of these roots.

Since $K_vT=b$, we have

$$K_v\tau = 0.29 \cdot a. \quad (43)$$

The gain is extremely small compared to the reference system, and the performance of the system will be poor. This can also be seen from the Nyquist plot of the uncompensated system in Fig. 5. It is evident that a considerable reduction in gain is required to ensure that $N=0$.

Type I Compensation

The system is augmented with the compensation matrix $C=\frac{1}{4}A^{-1}$. The augmented matrix B has four eigenvalues $\lambda_k (k=1, 2, 3, 4)$ [see (22)]. From (31) and (32), we have $h=\frac{1}{4}$ and $\lambda_k=\frac{1}{2}$. The poles of the closed loop system are the roots of

$$\frac{1}{G(s)} + \lambda_k = 0. \quad (44)$$

Substituting $G(s)$ and using $\zeta=0.6$ gives

$$K_v\tau = 1.89. \quad (45)$$

The velocity error matrix can be obtained from (68), in Appendix II, where $(AC)^{-1}=(\frac{1}{4}I)^{-1}=4I$. Hence,

$$\epsilon(\infty) = \frac{4}{K_v}U = 2.12\tau U, \quad (46)$$

where U is a single column unit matrix.

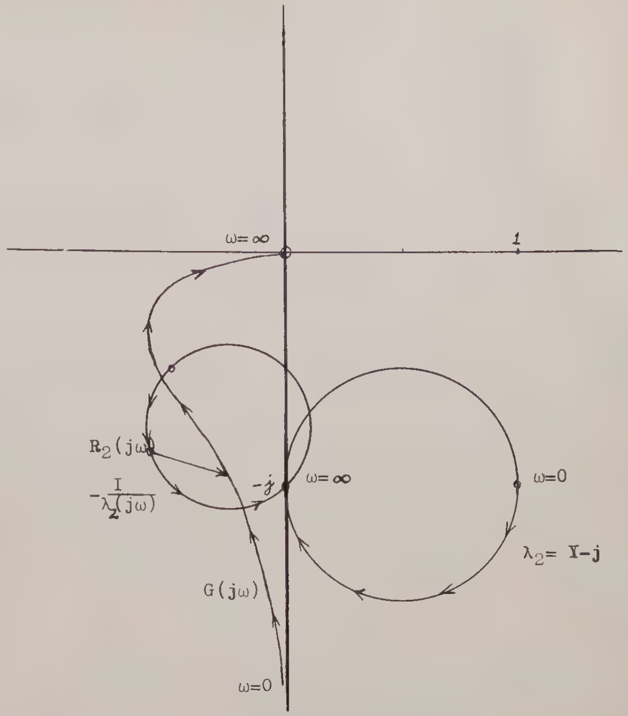

Fig. 5—λ-plane, uncompensated system.

The augmented system has the same relative damping ratio and double the velocity error of the reference system. System performance can be further improved by shaping the G-locus.

It is of interest to determine the elements of the C-matrix for type I compensation.

In this case,

$$4C = A^{-1} = \begin{pmatrix} \frac{\Delta_{11}}{\Delta} & \frac{\Delta_{21}}{\Delta} \\ \frac{\Delta_{12}}{\Delta} & \frac{\Delta_{22}}{\Delta} \end{pmatrix}, \quad (47)$$

where the Δ_{jk}'s are the cofactors of A and $\Delta=|A|$. We have

$$\begin{aligned} \frac{\Delta_{11}}{\Delta} &= \frac{Y}{1+Y^2} = \frac{\Delta_{22}}{\Delta}, \\ \frac{\Delta_{21}}{\Delta} &= \frac{Y}{1+Y^2} = -\frac{\Delta_{12}}{\Delta}, \end{aligned} \quad (48)$$

where $Y=1/(1+u)$ and $u=sT$.

Since the elements of C are to be realized by means of transfer functions, one possibility is to set [see Fig. 6(a)]

$$\frac{\Delta_{11}}{\Delta} = \frac{Z_1}{Z_1+Z_2} \equiv \frac{1}{1+\frac{Z_2}{Z_1}} = \frac{1}{1+\frac{u^2+u+1}{u+1}}. \quad (49)$$

Arbitrarily choosing $Z_2=1$ leads to [see Fig. 6(b)]

$$Z_1 = \frac{1}{u+\frac{1}{1+u}}. \quad (50)$$

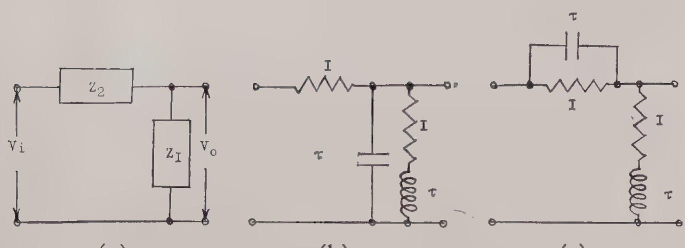

Fig. 6—Elements of the compensation matrix X.

Similarly,

$$\frac{\Delta_{21}}{\Delta} = \frac{1}{1 + \frac{Z_2}{Z_1}} = \frac{1}{1 + \frac{1}{(1+u)^2}}. \quad (51)$$

Arbitrarily choosing $Z_2 = 1/(1+u)$ leads to [see Fig. 6(c)]

$$Z_1 = 1 + u. \quad (52)$$

Type II Compensation

In this case, $C = A^T$ and hence

$$CA = A^T A = \begin{pmatrix} 1+Y^2 & 0 \\ 0 & 1+Y^2 \end{pmatrix} = (1+Y^2)I. \quad (53)$$

The eigenvalues for CA are $\gamma_1 = \gamma_2 = 1 + Y^2$. The eigenvalues $\lambda_k(s)$ for the augmented matrix B are determined from (29). Since

$$G(s) = \frac{K_v}{s(1+s\tau)} = \frac{b}{u(1+au)}, \quad (54)$$

and since a has been assumed small, u will be large in the region of interest ($u \cong 1/a$). This means that the eigenvalue locus $\lambda_k(j\omega)$ need not be completely determined, and a good approximation is obtained by simply considering the value $\lambda_k(j\infty)$. Since $Y(j\infty) = 0$, $\gamma_1 = \gamma_2 = 1$, and (29) gives

$$\lambda_k(j\infty) = \tfrac{1}{2}[1 \pm j\sqrt{3}], \quad (55)$$

the two points

$$-\frac{1}{\lambda_k}(j\infty) = -\tfrac{1}{2}[1 \mp j\sqrt{3}] \quad (56)$$

can then be considered as critical points, and conventional single variable techniques applied.

Appendix II

Error coefficients can be used as a performance index for a stable system. For a one variable system the error is

$$E_1 = \frac{1}{1+G(s)} V_1 = G^{-1}[1 + G^{-1}]^{-1} \cdot V_1. \quad (57)$$

The velocity error for a unit ramp input is

$$\epsilon_1(\infty) = \lim_{s \to 0} sE_1(s). \quad (58)$$

For zero position error we must have

$$\lim_{s \to 0} \frac{1}{G(s)} = 0; \quad (59)$$

hence

$$\epsilon_1(\infty) = \liminf_{s \to 0} \frac{G^{-1}(s)}{s} = \frac{1}{K_v}, \quad (60)$$

where K_v is the velocity error coefficient.

For the augmented multivariable servo system, the error matrix is readily determined from Fig. 3, and is given by

$$E = -\lambda W. \quad (61)$$

Eqs. (24) and (27) can be used to eliminate W and λ. The result is then

$$E = -\gamma C^{-1} X. \quad (62)$$

From (26) we obtain

$$X = (CA - \gamma I)^{-1} \cdot CV. \quad (63)$$

Using the identity

$$\begin{aligned}(CA - \gamma I)^{-1} C &= (I - \gamma A^{-1}C^{-1})^{-1} \cdot A^{-1}C^{-1}C \\ &= (I - \gamma A^{-1}C^{-1})^{-1} \cdot A^{-1}, \end{aligned} \quad (64)$$

and eliminating X from (62) and (63), it is easy to obtain

$$E = -\gamma C^{-1}A^{-1}(I - \gamma C^{-1}A^{-1})^{-1} \cdot V. \quad (65)$$

For zero position error we must have

$$\liminf_{s \to 0} -\gamma C^{-1}A^{-1} = 0. \quad (66)$$

Hence, for unit ramp inputs,

$$\epsilon(\infty) = \liminf_{s \to 0} -\frac{\gamma C^{-1}A^{-1}U}{s}, \quad (67)$$

where $\epsilon(\infty)$ is the velocity error matrix and U is a unit column matrix. Eqs. (65) and (67) correspond to (57) and (60) in the one variable case. Eq. (67) may be further simplified. From (16)

$$-\frac{\lambda}{s} = \frac{1}{sG(s)} \to \frac{1}{K_v} \text{ as } s \to 0.$$

Hence $\lambda \to 0$ and $\gamma \to \lambda$. Then

$$-\frac{\gamma}{s} \to -\frac{\lambda}{s} \to \frac{1}{K_v}.$$

The velocity error matrix is then

$$\epsilon(\infty) = \frac{1}{K_v} \cdot \liminf_{s \to 0} (AC)^{-1}U. \quad (68)$$

By a similar analysis it may be shown that (68) is also valid for the uncompensated system, provided C is replaced by I.

Appendix III

A few of the results from linear matrix—or vector—algebra are included for reference purposes:

The scalar product of two vectors u and v can be defined by

$$(u, v) = u_1^*v_1 + \cdots + u_n^*v_n = (v, u)^*, \tag{69}$$

where $u_1, \cdots, u_n$ are the elements of u, and u_k^* is the conjugate of u_k. Then

$$(u, u) = |u_1|^2 + \cdots + |u_n|^2 \geqq 0. \tag{70}$$

The adjoint B^* of a matrix B is defined by

$$(u, Bv) = (B^*u, v). \tag{71}$$

This definition is that used by Laning and Battin.[10] It is easily verified that $B^*(s) = B^T(s^*)$, where B^T is the transpose of B. If $B^* = B$, the matrix is hermitian:

[10] J. Laning and R. Battin, "Random Processes in Automatic Control," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 332–333; 1956.

Consider the eigenvalue equation

$$Bx = \lambda x; \tag{72}$$

scalar mutliplying by x gives

$$(x, Bx) = \lambda(x, x). \tag{73}$$

If B is hermitian, (x, Bx) is real, since

$$(x, Bx) = (B^*x, x) = (Bx, x) = (x, Bx)^*.$$

Hence

$$\lambda = \frac{(x, Bx)}{(x, x)} \geqq 0. \tag{74}$$

We can restrict ourselves to values $\geqq 0$, since, if $\lambda \leqq 0$, we can replace B by $-B$.

If

$$B = A^* \cdot A,$$

then

$$(Bx, x) = (A^*Ax, x) = (Ax, Ax) = (x, Bx) \geqq 0.$$

Hence B is hermitian and λ is real and >0.

Correspondence

Comments on "Optimization Based on a Square Error Criterion"*

The main results in the paper by Murphy and Bold[1] appear to be incorrect. The first one is (46) of the paper, which gives the optimum system frequency function as

$$G_{\text{opt}}(j\omega) = \frac{\Psi_{wc_dr}(j\omega, j\omega')}{\Psi_{wrr}(j\omega, j\omega')}. \quad (1)$$

This is the case where the question of physical realizability is disregarded. The functions Ψ_{wc_dr} and Ψ_{wrr} are the double Fourier transforms of the weighted correlation functions corresponding to the desired output $c_d(t)$ and the input $r(t)$. Since their result is based on minimizing the weighted square of the error

$$E = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} W(t)e^2(t)dt, \quad (2)$$

then, for the case where $W(t)=1$, their result should be reducible to the classical Wiener result for pure smoothing,

$$G_{\text{opt}}(j\omega) = \frac{S_{rc_d}(\omega)}{S_{rr}(j\omega)} = \frac{S_s(\omega)}{S_s(\omega)+S_N(\omega)}. \quad (3)$$

Consider the case of pure smoothing of a signal in the presence of white noise. Let the autocorrelation function of the signal input be

$$\phi_s(\tau) = e^{-|\tau|} = e^{-|x-y|}, \quad \text{for all } \tau, x, \text{ and } y. \quad (4)$$

The corresponding signal power spectrum is

$$S_s(\omega) = \int_{-\infty}^{0} e^{\tau}e^{-j\omega\tau}dt + \int_{0}^{\infty} e^{-\tau}e^{-j\omega\tau}dt$$
$$= \frac{2}{1+\omega^2}. \quad (5)$$

The double Fourier transform of $e^{-|x-y|}$ with respect to x and y, however, does not exist. We have

$$\int_{-\infty}^{\infty} e^{-|x-y|}e^{-j\omega x}dx$$
$$= \int_{-\infty}^{y} e^{x-y}e^{-j\omega x}dx + \int_{y}^{\infty} e^{-x+y}e^{-j\omega x}dx$$
$$= \frac{2e^{-j\omega y}}{1+\omega^2}. \quad (6)$$

But

$$\Psi_{rr}(j\omega, j\omega') = \int_{-\infty}^{\infty} \frac{2}{1+\omega^2}e^{-j\omega y}e^{-j\omega' y}dy \quad (7)$$

does not exist. Hence, (46) is meaningless for this example.

The root of the trouble seems to be in the step taken by the authors in going from (39) to (40). Their (39) is

$$\left\{\int_{-\infty}^{\infty} g(y)[\phi_{wrr}(\gamma - x, \tau - y) + \phi_{w_{rr}}(\tau - x, \gamma - y)]dy - \phi_{wc_dr}(\tau, \gamma - x) - \phi_{wc_dr}(\gamma, \tau - x)\right\}\Bigg|_{\tau=\gamma=0} = 0, \quad (8)$$

for x greater than zero. Note that the left-hand side of the above equation is independent of τ and γ. The authors claim that taking the double Fourier transform of both sides of their (39) yields their (40),

$$\left\{\int_{-\infty}^{\infty} e^{-j\omega\tau}\int_{-\infty}^{\infty} e^{-j\omega\gamma}\int_{-\infty}^{\infty} g(y)[\phi_{wrr}(\gamma-x, \tau-y) + \phi_{wrr}(\tau - x, \gamma - y)]dyd\gamma d\tau - \int_{-\infty}^{\infty} e^{-j\omega'\tau}\int_{-\infty}^{\infty} e^{-j\omega\gamma}[\phi_{wc_dr}(\tau, \gamma - x) + \phi_{wc_dr}(\gamma, \tau - x)]d\gamma dt\right\}\Bigg|_{\tau=\gamma=0}. \quad (9)$$

This is, of course, incorrect. Since the left-hand side of (8) is independent of τ and γ, the Fourier transform, with respect to τ and γ or both does not even exist. The rest of their equations leading to (46) are, therefore, questionable.

The principal contribution of the paper seems to be the derivation of their (39), which may be simplified further as

$$\int_{-\infty}^{\infty} g(y)[\phi_{rr}(-x, -y)]dy = \phi_{wc_dr}(0, -x), \quad \text{for } x > 0. \quad (10)$$

For physical realizability, $g(y)$ must be zero for y negative, so that (10) becomes

$$\int_{0}^{\infty} g(y)[\phi_{wrr}(-x, -y)]dy = \phi_{wc_dr}(0, -x), \quad \text{for } x > 0. \quad (11)$$

This is, of course, a generalized Wiener-Hopf integral equation. The weighted correlation functions are

$$\phi_{wrr}(t_1, t_2) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} W(t)r(t+t_1)r(t+t_2)dt \quad (12)$$

and

$$\phi_{wc_dr}(t_1, t_2) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} W(t)c_d(t+t_1)r(t+t_2)dt. \quad (13)$$

Note that for

$$W(t) = 1, \quad \text{for} \quad -\infty < t < \infty \quad (14)$$

(11) reduces to

$$\int_{0}^{\infty} g(y)\phi_{rr}(x-y)dy = \phi_{c_dr}(-x), \quad (15)$$

which is the usual Wiener-Hopf integral equation.

Discussions with Dr. M. E. Van Valkenburg are greatly appreciated.

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* Received by the PGAC, April 11, 1960.

[1] G. J. Murphy and N. T. Bold, "Optimization based on a square-error criterion with an arbitrary weighting function," IRE Trans. on Automatic Control, vol. AC-5, pp. 24–30; January, 1960.

Author's Comment[2]

I am pleased to note the interest of Cruz and his colleagues in the paper under discussion, and I am sorry to learn of their difficulty in understanding the material in the paper. However, I wish to point out that this difficulty was caused not by an error in the work described in the paper,[1] but rather by the use of a somewhat uncommon notation, which was employed, in spite of its shortcoming, for the purpose of reducing the complexity of (40) in the paper.

An alternative to the approach followed in the paper is to rewrite (39) in the form

$$\left[\mathfrak{F}_2^{-1}\mathfrak{F}_2\left\{\int_{-\infty}^{\infty} g(y)[\phi_{wrr}(\gamma - x, \tau - y) + \phi_{wrr}(\tau - x, \gamma - y)]dy - \phi_{wc_dr}(\tau, \gamma - x) - \phi_{wc_dr}(\gamma, \tau - x)\right\}\right]\Bigg|_{\tau=\gamma=0} = 0, \ x > 0, \quad (39a)$$

provided that the double Fourier transform denoted by $\mathfrak{F}_2\{\quad\}$ exists. An equivalent form is

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} e^{-j\omega\tau}\int_{-\infty}^{\infty} \cdot e^{-j\omega\gamma}\int_{-\infty}^{\infty} g(y)[\phi_{wrr}(\gamma - x, \tau - y) + \phi_{wrr}(\tau - x, \gamma - y)]dyd\gamma d\tau - \int_{-\infty}^{\infty} e^{-j\omega_1\tau}\int_{-\infty}^{\infty} e^{-j\omega\gamma}[\phi_{wc_dr}(\tau, \gamma-x) + \phi_{wc_dr}(\gamma, \tau - x)]d\gamma d\tau\right\}d\omega d\omega' = 0,$$
$$x > 0, \quad (39b)$$

provided again that the double Fourier transform exists. Following the steps outlined on page 28 of the paper leads from (39b) to (43) of the paper. Then, as is shown in the paper, the optimum frequency function is found to be given by (46) of the paper.

The example presented in (4)–(7) of Mr. Cruz's discussion is based on the assumption that $W(t)\equiv 1$. In this case, the two-space correlation functions reduce to the ordinary correlation functions, the problem reduces to the usual problem, and there is no need to consider double Fourier transforms at all. In any event, it is clearly stated in the paper that (40) follows from (39) only if the integral is absolutely convergent. Obviously, Mr. Cruz chose an example in which this condition is not met. Hence his criticism is not justified.

It should be noted that merely by choosing a sinusoidal form for $r(t)$ one could similarly "show" that the well-known solution to the ordinary Wiener-Hopf integral equation obtained through the use of either the Bode-Shannon method or the Wiener-

[2] Received by the PGAC, April 25, 1960.

Kolmogoroff method is "meaningless." Clearly, in this example as well as in the argument presented by Mr. Cruz, the fault lies in the choice of an input function for which the method is not rigorously applicable.

Alternatively, as is sometimes done in using the Wiener-Kolmogoroff method, one can resort to the concept of impulse functions, writing

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} d\omega = u_0(\tau). \tag{66}$$

Then (7) of Mr. Cruz's example becomes

$$\Psi_{wss}(i\omega, j\omega') = \frac{2}{1+\omega^2}\int_{-\infty}^{\infty} e^{-iy(\omega+\omega\ )}dy \tag{67}$$

$$= \frac{4\pi}{1+\omega^2} u_0(\omega+\omega'). \tag{68}$$

For the white noise one has

$$\phi_{nn}(\tau) = u_0(\tau) \tag{69}$$

$$\phi_{nn}(x-y) = u_{.0}(x-y) \tag{70}$$

and

$$\Psi_{nn}(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega x} u_0(x-y)dx \tag{71}$$

$$= e^{-j\omega y}. \tag{72}$$

Then

$$\mathfrak{F}\Psi_{nn}(j\omega) = \Psi_{wnn}(j\omega, j\omega') \tag{73}$$

$$= \int_{-\infty}^{\infty} e^{-j\omega\ y} e^{-j\omega y} dy \tag{74}$$

$$= 2\pi u_0(\omega+\omega'). \tag{75}$$

It follows from (49) that

$$G_{\mathrm{opt}}(j\omega) = \frac{G_d(j\omega)\left[\dfrac{4\pi}{1+\omega^2}u_0(\omega+\omega')\right]}{\dfrac{4\pi}{1+\omega^2}u_0(\omega+\omega') + 2\pi u_0(\omega+\omega')} \tag{76}$$

$$= \frac{\dfrac{2}{1+\omega^2}}{\dfrac{2}{1+\omega^2}+1} G_d(j\omega). \tag{77}$$

Using the usual (Wiener) approach, one has

$$G_{\mathrm{opt}}(j\omega) = \frac{G_d(j\omega)\Psi_{ss}(j\omega)}{\Psi_{ss}(j\omega)+\Psi_{nn}(j\omega)} \tag{78}$$

$$= \frac{\dfrac{2}{1+\omega^2}}{\dfrac{2}{1+\omega^2}+1} G_d(j\omega). \tag{79}$$

Obviously, the result obtained in (77) is identical to that obtained in (79). Thus, it has been shown that, contrary to Mr. Cruz's statement for the example chosen by him, the result obtained by application of the method presented in the paper is "reducible to the classical Wiener result."

I fail to see any point in the inclusion of (12) and (13) of the discussion, since these equations are identical to (26) and (25), respectively, of the paper.

The statement that "for

$$W(t) \equiv 1, \quad \text{for} \quad -\infty < t < \infty$$

(11) reduces to

$$\int_0^{\infty} g(y)\phi_{rr}(x-y)dy = \phi_{c_d r}(-x),$$

which is the usual Wiener-Hopf integral equation," was implied by (27) through (30) and the associated text in the paper.

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Nonlinear Feedback in Servo Systems*

In addition to having a high degree of stability, a good servo system should have a quick response and no overshoot. But these two requirements are more or less contradictory. A simple analysis of a second order servo system shows that in order to have quick response, the frictional damping must be kept to a low value. But a low damping usually gives a large overshoot which can only be minimized by increasing the damping.

Lewis[1] in his pioneering work showed that with a step input and a nonlinear feedback proportional to the product of the "error" and the rate of change of output, it is possible to solve the problem. It was shown that such a nonlinear feedback results in an effective damping which is small at the beginning but becomes large as the output approaches its final value. A low damping at the start gives a quick response. Again, no overshoot can occur, as the damping increases to a sufficiently large value before the output reaches its steady state level.

The above type of feedback arrangement, which produces a very good result for a step input (for which it was analyzed), is not very effective if the input is of any other type. In the feedback arrangement of Lewis, the increase in damping is actually proportional to "error," *i.e.*, difference between the input and the output. But as the input is constant for a "step," the effective increase in damping becomes proportional to the output in this particular case. Thus, we get an increased damping as the output approaches its steady state value and no overshoot occurs. If, on the other hand, the input is not a constant step but itself changes with time as in a "ramp" or a "trapezoid" (step with finite slope), the effective increase in damping (as the output approaches its final value) will normally be small and overshoots will occur.

If, however, we modify the system slightly by providing feedback signal proportional to the product of the output response and its time rate of change, better response is obtained for other types of inputs as well. Such a feedback arrangement is discussed in the following paragraphs.

In the servo arrangement discussed below, "$G(s)$" is the transfer function of the controller unit, and K_0 is a constant giving amplifier gain; x_i is the input signal to the system, and x_o is the resultant output. The first derivative (dx_o/dt) of the output (x_o)—as obtained from a rate generator—is multiplied with $K_0 x_o$ in a multiplier unit, and the product is added to the "error" $e(=x_i-x_o)$ in an adder. The resultant signal is then applied to the input of the controller unit and acts as the actuating signal for the controller. The arrangement is shown schematically in Fig. 1. Thus, the feedback signal is composed of the sum of the error signal and the product of output with its time rate of change.

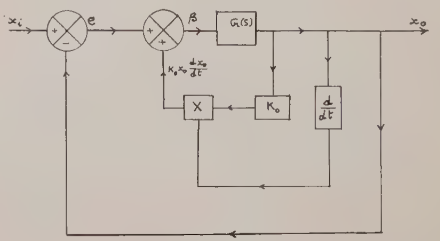

Fig. 1—Block diagram of the feedback arrangement.

The differential equation of such a feedback system may be represented by an equation of the type

$$J\frac{d^2x_o}{dt^2} + (R_0 + Bx_0)\frac{dx_o}{dt} + D(x_o + x_i) = 0,$$

where J and D are system constants, depending respectively on the inertia and the compliance of the system. R_0 is the frictional damping coefficient and B is a constant proportional to K_0. Or, we may write,

$$J\frac{d^2x_o}{dt^2} + R\frac{dx_o}{dt} + Dx_o = Dx_i \quad \cdots \tag{1}$$

where

$$R = R_0 + Bx_o \cdots . \tag{1a}$$

Now, if we regard R as a constant over a small increment of time, for which the change in x_o may be neglected, then the system can be treated as linear over that interval. Thus, the actual system is approximated as a series of linear systems with increasing values of R.

If $R = R_n(=R_0 + Bx_{on})$ be the value of R during the nth increment of time, then during that particular interval

$$J\frac{d^2x_{on}}{dt^2} + R_n\frac{dx_{on}}{dt} + Dx_{on} = Dx_i(t_n) \quad \cdots \tag{2}$$

where x_{on} is the value of x_o at the nth interval.

It is to be noted that (2) is identical with the equation derived by Lewis[1] if the input signal happens to be a step function (*i.e.*, x_i is equal to a constant). In that case, the above feedback arrangement will give identical response with that of Lewis. The effective value of R_n in the Lewis system is given by

$$R_n = R_0' + B(x_{on} - x_{in})$$

* Received by the PGAC, April 5, 1960.

[1] J. B. Lewis, *Trans. AIEE*, vol. 71, p. 449; 1953.

where x_{in} is the value of x_i at the nth instant. The above expression for R_n becomes identical with that of (1a) for the system under discussion only when x_i is a constant, *i.e.*, for a step input. But if x_i varies with time (as for all other types of inputs), the effective damping in the above system will be different from that of Lewis and is found to give a better response as shown later.

Consider for example, a unit ramp input, for which $x_i = t$. In that case, (2) becomes

$$J\frac{d^2x_{on}}{dt^2} + R_n\frac{dx_{on}}{dt} + Dx_{on} = Dt.$$

Solving this equation with the help of Laplace transform, we get

$$x_{on} = t - \frac{J}{D}\left[\frac{R_n}{J} + \frac{\epsilon^{-(R_n/2J)t}}{\sqrt{\frac{D}{J} - \frac{R_n^2}{4J^2}}} \cdot \left\{\left(\frac{R_n^2}{2J^2} - \frac{D}{J}\right)\sin\left(\sqrt{\frac{D}{J} - \frac{R_n^2}{4J^2}}\right)t + \frac{R_n}{J}\sqrt{\frac{D}{J} - \frac{R_n^2}{4J^2}} \cdot \cos\left(\sqrt{\frac{D}{J} - \frac{R_n^2}{4J^2}}\right)t\right\}\right]\cdots \quad (3)$$

In (3), $\sqrt{D/J}$ is the undamped natural frequency of the system and we may put $\sqrt{D/J} = \omega_0$. Also, for the case of critical damping, we have

$$(R_n)_c = \sqrt{\frac{D}{J}\cdot 4J^2} = 2\sqrt{JD}\cdots \quad (4)$$

where $(R_n)_c$ is the value of R_n for the case of critical damping.

It is evident from (1a) that the value of R_n will increase with time as the output increases and the sinusoidal (and cosinusoidal) terms in (3) will die down rapidly with time, thus giving the steady state output of $(t - R_n/D)$ in a short while. But, due to low damping at the initial stage, the output will more closely follow the input than for the critically damped (or overdamped) case. A typical example is calculated and shown below.

For the feedback arrangement under consideration, values of output response for the nth interval of time [as given by (3)] have been calculated successively by the method of numerical computation and successive approximation, as was also done by Lewis. In the example chosen to illustrate the result, the value of initial damping R_0 was taken as $0.1(R_n)_c$ and the value of B was chosen as $0.9(R_n)_c$. Both the time (t) and the output (x_o) have been calculated and plotted in terms of ω_0. To calculate the value of R_n we have taken ω_0 as 6 radians per second. Any other value of ω_0 would have given similar results. The calculated variation of output with time for this typical case has been plotted in curve e of Fig. 2. In the same figure are also shown the characteristic curves for the input (curve a), the output with zero damping (curve b), and the output with critical damping (curve c). As expected, with zero damping, the output is oscillatory with large overshoots and with critical damping there is an appreciable time delay between the input signal and the output response. Both these results are unsatisfactory for most applications. But with the feedback arrangement shown schematically in Fig. 1, there is no oscillation and no time delay (as shown by curve e). The overshoot is also negligibly small. Steady state error is also found to be less than that under the critically damped condition.

The output response with Lewis's feedback arrangement has also been plotted for comparison as curve d on the same figure (Fig. 2). Since, for the underdamped case, the error is never large and is actually zero at some points, there is no effective increase in damping with time. The output maintains its damped oscillatory nature and appreciable overshoots occur. Thus, we see that with the feedback arrangement depicted in Fig. 1, a step input gives the same type of improved response as that of Lewis, whereas for a ramp input a better response is obtained.

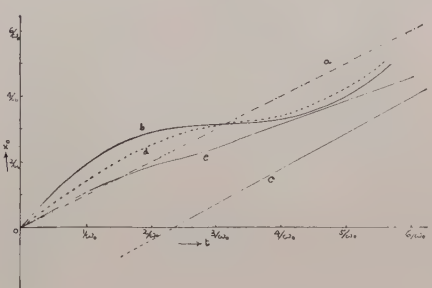

Fig. 2—Output responses for the given system: curve b—output with zero damping; curve c—output with critical damping; curve d—output with Lewis' feedback arrangement; curve e—output with the feedback arrangement under discussion (shown schematically in Fig. 1). The input ramp is shown as curve a for comparison.

It may be mentioned that improved response occurs with the above feedback arrangement only if the initial damping of the system is low. But this is not a serious handicap. If the system is initially overdamped or critically damped, we can reduce the effective damping of the system by providing an additional feedback loop with a suitable amount of positive feedback.

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Connection Between Various Methods of Investigating Absolute Stability*

A general problem of control may be formulated as follows: Given a system (Fig. 1) where $f(\sigma)$ may be any nonlinearity from the class defined by

$$f(\sigma) = 0, \sigma = 0; \quad \sigma f(\sigma) > 0, \quad \sigma \neq 0, \quad (1)$$

what are the restrictions that must be placed on G to ensure absolute stability? Lure[1] discusses this problem in considerable detail and, using Lyapunof's "second method," derives after lengthy calculations rigorous stability criteria which guarantee absolute stability for any $f(\sigma)$ as defined above. These criteria and others derived by Lyapunef's "second method"[2] are difficult to apply to complicated systems, and physical insight into the underlying reasons for stability is often lost because of the cumbersome calculations.

Another approach which clarifies this problem is given by Sobolev,[3] who compares the nonlinear system (Fig. 1) to a linear system (Fig. 2). He proves the following theorem:

If the linear system is stable for all $0 < h < \infty$, then the nonlinear system is absolutely stable. Thus, absolute stability of the nonlinear system is given by the requirement for unconditional stability of the associated linear system. Hence, the equation

$$1 + hG = 0, \quad 0 < h < \infty \quad (2)$$

must have no roots with positive real parts.

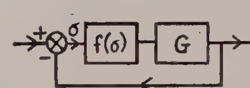

Fig. 1.

Fig. 2.

Sobolev solves this equation by using the familiar Routh-Hurwitz criteria. The resulting equations define the stability region of the linear system with h as parameter. The envelope of this function gives the stability region of the linear system for all h, and hence by the above theorem the region of absolute stability of the nonlinear system.

The above stability condition (2) can also be written as

$$G = -\frac{1}{h}, \quad 0 < h < \infty. \quad (3)$$

Hence, the stability region for the nonlinear system is given by

$$\underline{/G(j\omega)} > -180°. \quad (4)$$

In this form, the significance of Sobolev's theorem is made clearer. Not only because the calculations are simpler, obviating the need for finding the envelope, but also,

* Received by the PGAC, June 10, 1960.

[1] A. I. Lure, "Some Non-Linear Problems in the Theory of Control," H. M. Stationary Office, London; 1957.

[2] A. K. Bedel'baev, "Some simplified stability criteria for non-linear controlled systems," *Automat. i Telemekh.*, vol. 20, pp. 689–701; June, 1959.

[3] Yu. S. Sobolev, "The absolute stabilities of certain controlled systems," *Automat. i Telemekh.*, vol. 20, pp. 401–405; April, 1959.

mainly, because the role of the transfer function in determining the stability of the nonlinear system is placed very much in evidence.

Eq. (3) has a striking resemblance to the equation for determining stability by the describing function method.

$$G = -\frac{1}{N(\sigma)}, \tag{5}$$

where $N(\sigma)$ is the describing function for the nonlinearity $f(\sigma)$. By the above assumptions on $f(\sigma)$ (1), $N(\sigma)$ is real and positive. Hence, by describing function analysis, stability in the large will again be ensured only if $\underline{/G(j\omega)} > -180°$ for all ω. Thus, the approximate describing function method yields the same stability criterion as given by Sobolev's rigorous approach.

Fig. (4) also enables a simpler approach to the calculation of the boundaries of the stability region. Consider Sobolev's first example (Fig. 3).

The transfer function is

$$G = \frac{1}{s(W + T_i^2 s)} \cdot \left[r + \frac{N(a + Es + G^2 s^2)}{s(U + T_a^2 s)} \right]$$

$$= \frac{1}{s^2} \cdot \frac{s^2(NG^2 + Ta^2 r) + s(ur + NE) + Na}{s^2 T_i^2 T_a^2 + s(uT_i^2 + WT_a^2) + uW}$$

$$= \frac{1}{s^2} \cdot \frac{s^2 M_2 + sM_3 + N_a}{s^2 T_i^2 T_a^2 + sM_1 + UW}$$

$$= \frac{1}{s^2} \cdot \frac{P(s)}{Q(s)}.$$

The stability boundary is given by

$$\underline{/G(j\omega)} = -180°. \tag{6}$$

Hence,

$$\underline{/P(j\omega)} - \underline{/Q(j\omega)} = 0,$$

or

$$\tan^{-1} \frac{\omega M_3}{Na - \omega^2 M_2} - \tan^{-1} \frac{\omega M_1}{uW - \omega^2 T_i^2 T_a^2} = 0.$$

After some algebra, we arrive at

$$\omega = \sqrt{\frac{uWM_3 - N_a \cdot M_1}{T_i^2 T_a^2 M_3 - M_2 M_1}}.$$

The system will be stable only if this equation has no real roots for all ω. The stability region is shown in Fig. 4. Substituting for M_1, M_2, and M_3, we arrive at Sobolev's conditions:

$$UW(ur + NE) - Na(uT_i^2 + WT_a^2) \geq 0$$

$$NT_i^2(ET_a^2 - G^2 u) - (NG^2 + T_a^2 r)WT_a^2 \leq 0.$$

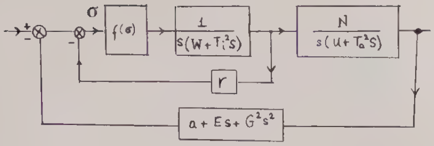

Fig. 3.

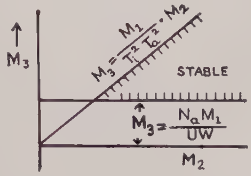

Fig. 4.

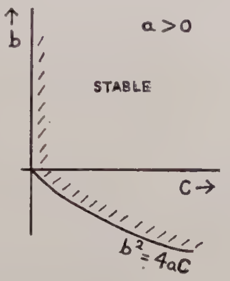

Fig. 5.

As a second example, consider Lure's general canonical form,

$$\dot{Z}_\rho = \lambda_\rho Z_\rho + f(\sigma), \qquad \rho = 1, 2, \cdots, n$$

$$\dot{\sigma} = \sum_{\rho=1}^{n} \beta_\rho Z_\rho - rf(\sigma). \tag{7}$$

By eliminating all Z_ρ's, we obtain the open loop transfer function

$$G = \frac{1}{S}\left[r + \sum_{\rho=1}^{n} \frac{\lambda_\rho}{\lambda_\rho - S} \right].$$

For $n=2$, the transfer function becomes

$$G = \frac{1}{s} \cdot \frac{s^2 r - s[r(\lambda_1 + \lambda_2) + \beta_1 + \beta_2] + r\lambda_1\lambda_2 + \beta_1\lambda_2 + \beta_2\lambda_1}{s^2 - s(\lambda_1 + \lambda_2) + \lambda_1\lambda_2}$$

applying again the stability boundary criterion $G(j\omega) = 180°$, we arrive after some algebra at the bi-quadratic equation

$$\omega^4 r + \omega^2[r(\lambda_1^2 + \lambda_2^2) + \beta_1\lambda_1 + \beta_2\lambda_2]$$

$$+ \lambda_1^2\lambda_2^2 \left[r + \frac{\beta_1}{\lambda_1} + \frac{\beta_2}{\lambda_2} \right] = \omega^4 \cdot a + \omega^2 \cdot b + c$$

$$= 0.$$

The stability region in this case is shown in Fig. 5.

One boundary is given by $c \geqq 0$, or

$$\Gamma^2 = r + \frac{\beta_1}{\lambda 1} + \frac{\beta_2}{\lambda 2} \geq 0 (\lambda_1, \lambda_2 \neq 0). \tag{8}$$

This is Lur'e's first condition.[4] The second boundary is given by the lower branch of the parabola $b^2 = 4ac$, or

$$[r(\lambda_1^2 + \lambda_2^2) + \beta_1\lambda_1 + \beta_2\lambda_2]^2 = 4r\lambda_1^2\lambda_2^2\Gamma^2. \tag{9}$$

Lure's second condition is[4]

$$\pi^2 = rf_1^2 - 2rf_2 + B_1 + 2\epsilon\sqrt{r}\Gamma f_2 = 0. \tag{10}$$

Upon substituting

$$\epsilon = \pm 1; \qquad B_1 = \beta_1\lambda_1 + \beta_2\lambda_2;$$
$$f_1 = \lambda_1 + \lambda_2; \qquad f_2 = \lambda_1\lambda_2$$

We see that conditions (9) and (10) are the same. Hence, the simple criterion $G(j\omega) > -180°$ leads to the same results as obtained by Lyapunof's second method.

In conclusion,[5] Sobolev's comparison theorem leads to a simple and rigorous stability criterion. Also, it gives a firm basis to stability analysis by the describing function method.

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[4] Lure, *op. cit.*, p. 71, (9.6).

[5] *Note added in proof:* After writing this correspondence the author was told during the first IFAC Congress in Moscow that Sobolev's theorem is not always valid for $n > 2$. A lively correspondence on this subject has appeared in recent issues of *Avtomatika i Telemekhanika*. It has not yet been possible to obtain an English translation of these issues. However, the results of the above examples are correct and agree with those derived by Lyanimof's second method.

On the Approximation of Roots of *n*th Order Polynomials*

The roots of an *n*th order polynomial may be obtained by a wide variety of techniques. Lin's method[1] of quadratic factor extraction is particularly useful, because of its simplicity, when one is dealing with polynomials having real coefficients where it is known that complex roots will occur in conjugate pairs. While Lin's method converges to one of the quadratic factors of the polynomial, convergence is sometimes slow and, therefore, hand computation of a reasonable approximation can be tedious.

The root-locus method[2] provides an approximation which may then be used as a first trial factor in Lin's method for final convergence. This technique requires that the polynomial be rearranged into the familiar root-locus form,

$$1 + \frac{KN(s)}{D(s)} = 0. \tag{1}$$

One convenient way of accomplishing the rearrangement is to divide the polynomial by all but the three lowest order terms as indicated as follows:

* Received by the PGAC, June 1, 1960. The development discussed here was accomplished while the author was employed by the Microwave Res. Inst., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

[1] S. N. Lin, "A method of successive approximations of evaluating the real and complex roots of cubic and higher order equations," *J. Math. Phys.*, vol. 20, pp. 231–242; August, 1941.

[2] J. G. Truxal, "Root locus methods," in "Automatic Feedback Control System Synthesis," pp. 267–272, McGraw-Hill Book Co., Inc., New York, N. Y.; 1955.

$$P(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-2}s^2 + a_{n-1}s + a_n = 0 \tag{2}$$

$$0 = 1 + \frac{a_{n-2}\left(s^2 + \frac{a_{n-1}}{a_{n-2}} s + \frac{a_n}{a_{n-2}}\right)}{s^3(s^{n-3} + a_1 s^{n-4} + a_2 s^{n-5} + \cdots + a_{n-3})} \tag{3}$$

$$0 = 1 + \frac{a_{n-2}(s^2 + a_{n-1}' s + a_n')}{s^3(s^{n-3} + a_1 s^{n-4} + a_2 s^{n-5} + \cdots + a_{n-3})}. \tag{4}$$

If a_{n-2} is now considered a variable, (4) is of the same form as (1). With n less than six, factors of $N(s)$ and $D(s)$ will, at most, be quadratics and their roots may be easily determined, thus locating pole-zero positions. In such a case, the locus of pole positions as K varies may be plotted directly. The locations of poles for $K = a_{n-2}$ are the n roots of the original polynomial in s. With n six or more, a repeated application of this technique is suggested. That is to say,

$$D(s) = s^3(s^{n-3} + a_1 s^{n-4} + \cdots + a_{n-3}) \tag{5}$$

is set equal to zero and its roots are determined by means of the root-locus technique.

A useful extension of the root-locus technique is obtained if one notes that more than one representation for the root locus can be obtained from the same polynomial. If both loci are plotted, they must intersect at the roots of the original polynomial. Consider, for example, the following fourth order polynomial:

$$s^4 + as^3 + bs^2 + cs + d = 0. \tag{6}$$

Dividing by all but the three lowest order terms yields

$$0 = 1 + \frac{b\left(s^2 + \frac{c}{b} s + \frac{d}{b}\right)}{s^3(s + a)}. \tag{7}$$

Division by all but the two lowest order terms yields

$$0 = 1 + \frac{c\left(s + \frac{d}{c}\right)}{s^2(s^2 + as + b)}. \tag{8}$$

The intersecting root-loci technique described in these notes may be applied to polynomials of order 5 or greater by repeated application of the method to reduce denominator factors to quadratics.

It should be noted that loci will intersect over a range of values for real roots and, in general, at a point only for complex roots. Given such a range of values for real roots, it is often possible to reduce the order of the polynomial by trial-and-error removal of the real roots or by means of Lin's method applied to single real roots.

By way of illustration, consider the fourth-order polynomial

$$P(s) = s^4 + 15s^3 + 55s^2 + 245s + 204 = 0. \tag{9}$$

Arranging the polynomial in a root-locus form,

$$1 + \frac{55\left(s^2 + \frac{245}{55} s + \frac{204}{55}\right)}{s^3(s + 15)} = 0, \tag{10}$$

which reduces to

$$1 + \frac{K_1(s + 1.09)(s + 3.41)}{s^3(s + 15)} = 0. \tag{11}$$

A second root locus form is obtained as follows:

$$1 + \frac{245\left(s + \frac{204}{245}\right)}{s^2(s^2 + 15s + 55)} = 0, \tag{12}$$

$$1 + \frac{K_2(s + 0.83)}{s^2(s + 6.4)(s + 8.6)} = 0. \tag{13}$$

The loci for variable K in (11) and (13) are sketched in Figs. 1 and 2. Fig. 3 is a sketch of the intersections of the two loci, and thus indicates the roots of the polynomial of (9).

Examination of Fig. 3 indicates a real root in the range $-0.83 > s > -1.09$. One might well assume the root to be located at $s = -1$ (as, indeed, it is) and thereby reduce the polynomial. The reduced polynomial,

$$\frac{P(s)}{(s + 1)} = s^3 + 14s^2 + 41s + 204 = 0, \tag{14}$$

may be written in three root-loci forms:

$$1 + \frac{K_1(s + 4.98)}{s^2(s + 14)} = 0; \tag{15}$$

$$1 + \frac{K_2(s + 1.87 + j3.48)(s + 1.87 - j3.48)}{s^3}; \tag{16}$$

and

$$1 + \frac{K_3}{s(s + 4.14)(s + 9.86)} = 0. \tag{17}$$

A sketch the intersections of these loci is presented in Fig. 4.

On comparison of the intersecting root-locus technique with Lin's method of quadratic factor extraction, it is noted that, for the polynomial of (9), an oscillatory solution results from Lin's method, while Fig. 3

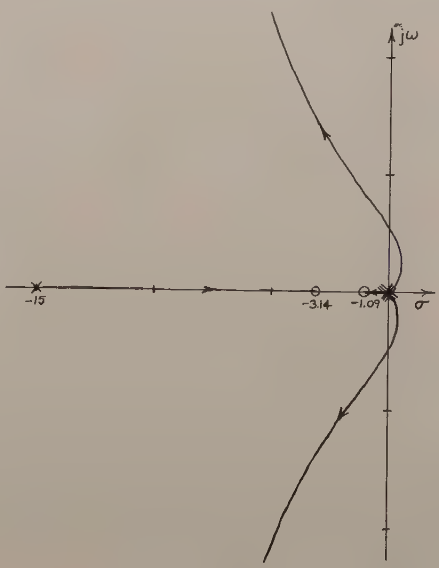

Fig. 1—Root locus sketch for $1 + \frac{K_1(s + 1.09)(s + 3.41)}{s^3(s + 15)} = 0.$

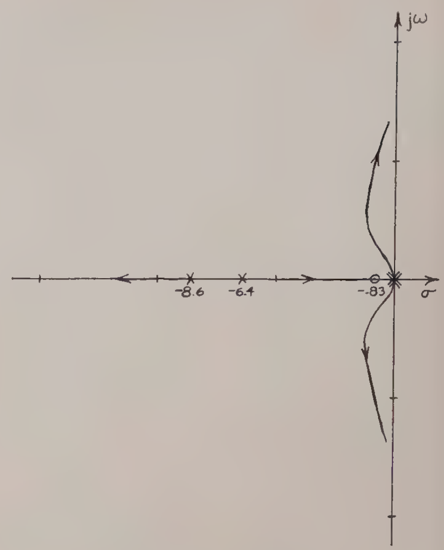

Fig. 2—Root locus sketch for $1 + \frac{K_2(s + 0.83)}{s^2(s + 6.4)(s + 8.6)} = 0.$

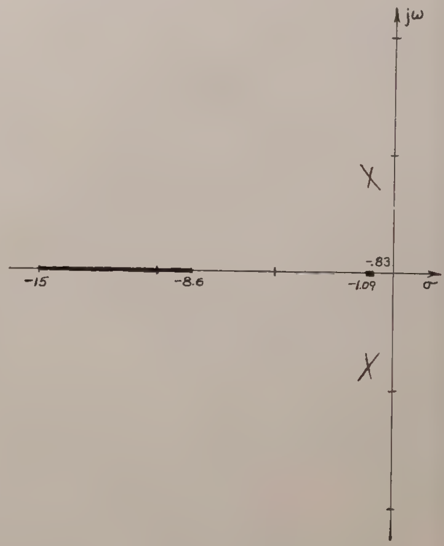

Fig. 3—Intersecting loci sketch for $P(s) = s^4 + 15s^3 + 55s^2 + 245s + 204 = 0$ $= (s + 1)(s + 12)(s^2 + 2s + 17) = 0.$

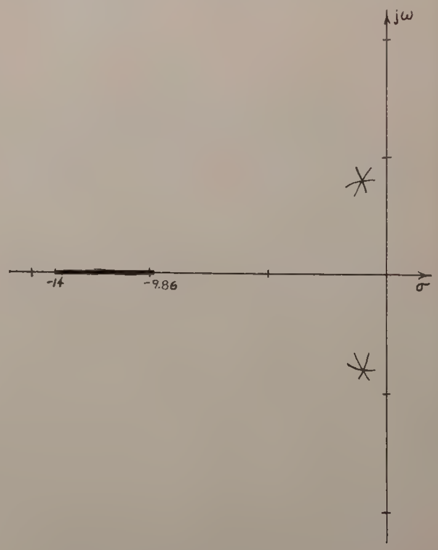

Fig. 4—Intersecting loci sketch for $P(s) = s^3 + 14s^2 + 41s + 204 = 0$ $= (s + 12)(s^2 + 2s + 17) = 0.$

yields a close approximation to the complex roots and one of the real roots. If one applies Lin's method to remove a real root from (9), convergence to the factor $(s+1)$ is obtained in three trials and the polynomial is reduced by an order. Reapplying Lin's method to obtain the complex roots (or the remaining real root), the complex roots are approximated by the third or fourth trial divisor.

The described extension enhances the value of the basic technique by indicating the regions of the roots from crude sketches of the root-loci. Increased accuracy may be obtained by any of three methods:

1) Both loci may be plotted carefully in the regions of intersection.
2) One of the loci may be plotted accurately in the regions of intersections and the points corresponding to K located on the locus.
3) The approximate intersection obtained from the crude sketch may be used as a first trial factor in another technique such as Lin's or Newton's method.

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Determination of Transfer Function Coefficients of a Linear Dynamic System from Frequency Response Characteristics*

Determination of transfer functions in standard form, from experimentally obtained frequency response characteristics, is a classical problem of extraordinary interest to designers of servo-systems. Levy[1] has recently presented a very versatile method which facilitates exact synthesis of frequency dependent systems. The method can be used to fix up an algebraic expression as the ratio of two polynomials, which could later be solved, with the aid of an Isograph,[2] to give the zeroes and poles of the system under study.

However, as pointed out by Levy himself, the utility of his method is limited by two major restrictions. The object of this correspondence is to suggest means of overcoming these restrictions in a very direct and elegant manner. The first restriction according to Levy is that systems with infinite gain at zero frequency cannot be handled by his method except by an indirect approach. These sytems are the familiar type I, type II, and type III Servo-systems. It is interesting to observe that this restriction is totally removed if one uses M-curves instead of G-curves. G and M refer to the open and closed-loop responses in the frequency domain, as per standard notation. M-curves can be easily obtained from the G-curves with the aid of a Nichols Chart. The use of M-curves, in preference to G-curves, results in several additional advantages. Firstly, in the case of unity feedback systems the denominator polynomial of the M-curve directly gives the characteristic equation of the system, which is useful for determining quantitatively the performance characteristic of the system in the time domain. Secondly, computation with M-curves is much easier because of the smooth nature of these curves. In any event, if one wishes to get the transfer function for G, it can be obtained as follows.

Let

$$M(j\omega) = \frac{f_1(j\omega)}{f_2(j\omega)}, \tag{1}$$

where f_1 and f_2 are polynomials in $j\omega$. As per standard notation,

$$M(j\omega) = \frac{G(j\omega)}{1+G(j\omega)}. \tag{2}$$

From (1) and (2),

$$G(j\omega) = \frac{f_1(j\omega)}{f_2(j\omega) - f_1(j\omega)}. \tag{3}$$

The necessity for using M-curves in case of type I, type II or type III systems can be illustrated by a specific example.

Let

$$G(j\omega) = f(j\omega)\cdot\frac{1}{(j\omega)^n}, \quad \text{for } n = 1, 2, 3, \cdots, \tag{4}$$

where $f(j\omega)$ is of type

$$\frac{A_0 + A_1(j\omega) + A_2(j\omega)^2 + \cdots}{1 + B_1(j\omega) + B_2(j\omega)^2 + \cdots}. \tag{5}$$

Eq. (4) obviously gives infinite gain at zero frequency. Considering the corresponding M-function, one gets from (2) and (4),

$$M(j\omega) = \frac{f(j\omega)}{(j\omega)^n + f(j\omega)}. \tag{6}$$

Eq. (6) gives a finite gain even at zero frequency, thus overcoming the first restriction.

The second restriction of Levy's method pertains to the weighting of the function for the error criterion. The use of the type of weighted error function suggested by Levy implies that for a given value of ω, the magnitude of the error is directly proportional to the magnitude of the function. In order to minimize the errors in computation, Levy sugggests selecting a greater number of sampling points in the critical region of the curve. A more direct means of overcoming this restriction is by working in terms of the inverse plots. By working with the inverse function, the weighting is such as to make the error a minimum in the neighborhood of the local maxima of the given function.

Let

$$M(j\omega) = \frac{n(j\omega)}{d(j\omega)}, \tag{7}$$

be the required function, where $n(j\omega)$ is the numerator and $d(j\omega)$ the denominator of $M(j\omega)$. Let

$$H(j\omega) = R(\omega) + JI(\omega) \tag{8}$$

be the experimental curve, with $R(\omega)$ denoting its real part, and $I(\omega)$ its imaginary part.

Let the numerical difference between the two functions $1/M(j\omega)$ and $1/H(j\omega)$ represent the error in curve-fitting, that is

$$\delta(\omega) = \frac{1}{H(j\omega)} - \frac{d(\omega)}{n(\omega)}. \tag{9}$$

Multiplying both sides of (9) by $n(\omega)$, one gets

$$\delta(\omega)\cdot n(\omega) = \frac{n(\omega)}{H(j\omega)} - d(\omega). \tag{10}$$

The RHS of (10) is a function of real and imaginary terms, which may be separated to give

$$\delta(\omega)\cdot n(\omega) = a(\omega) + jb(\omega); \tag{11}$$

or at any specific value of frequency $\omega = \omega_K$,

$$[\delta(\omega)\cdot n(\omega)]^2_{\omega=\omega K} = a^2(\omega_K) + b^2(\omega_K). \tag{12}$$

Now, the weighted error E is defined as the function given in (9) summed over the sampling freqencies ω_K. Thus,

$$E = \sum_{K=0}^{m} [a^2(\omega_K) + b^2(\omega_K)]. \tag{13}$$

The unknown coefficients are determined on the basis of minimizing the function E. By the above derivation, it is clear that $[\delta(\omega)\cdot n(\omega)]^2$ is a minimum, say K^2, for that set of coefficients determined by the principle of least squares of the weighted error function. For this choice, $\delta(\omega)$ is a minimum $K/n(\omega)$, where $n(\omega)$, or the numerator of $M(j\omega)$ is a maximum. Thus, the error in fitting is a minimum in the neighborhood of local maxima of M, an aspect of obvious advantage for design purposes. It is easily seen that the suggested modifications widen the scope of Levy's method to a considerable extent without introducing any new errors.

Further, the constant term of the numerator polynomial is equal to the magnitude of the function at negligibly small values of ω. Utilizing this data reduces the order of the matrix equation by one.

Recently, Dudnikov[3] has suggested an interesting method of determining the coefficients of the polynomials from the initial portion of an experimentally obtained amplitude-phase characteristic.

The transformer analog principle[4] can be advantageously applied for developing a special purpose computer to mechanize the the evaluation of the manipulated parameters λ, S, T and U. It is proposed to use this equipment in conjunction with a Mallock's equation solver,[5] to determine straightaway the coefficients of the numerator and denominator polynomials. Work on the development and design of this special purpose analog computer is under way.

The authors wish to express their sincere thanks to Prof. H. N. Ramachandra Rao for his kind interest in this work.

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* Received by the PGAC, June 8, 1960.

[1] E. C. Levy, "Complex curve fitting," IRE Trans. on Automatic Control, vol. AC-4, pp. 37–43; May, 1959.

[2] P. Venkata Rao, "A novel type of isograph," IRE Trans. on Electronic Computers, vol. EC-7, pp. 97–103; June, 1958.

[3] E. E. Dudnikov, "Determination of transfer function coefficients of a linear system," *Automation and Remote Control* (English translation of *Avtomat. i Telemekh.*), vol. 20, pp. 552–559; May, 1959.

[4] P. Venkata Rao and G. Krishna, "The transformer analogue computer (TAC)," *Trans. AIEE*, vol. 34, pp. 732–739; January, 1958.

[5] M. W. Humphrey Davies and G. R. Slemon, "Transformer analogue network analyzer," *Proc. IEE*, vol. 100, pt. 2, pp. 469–486; 1953.

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From 1951 to 1956, as project engineer and senior engineer at the Sperry Gyroscope Company, Great Neck, N. Y., he worked on various developments in the field of aircraft flight control, including an inertial navigation system, and the automatic pilot systems for the B-47, B-52 and C-130 aircraft. Since 1958, he has been with the Sperry Phoenix Company, a division of Sperry-Rand Corp., Phoenix Ariz., where he is engineering supervisor in charge of the Advanced Systems Engineering Section of the Flight Controls Engineering Department. His present work includes supervising projects aimed at the development of automatic landing systems, adaptive flight control systems, and the application of digital computer techniques to advanced flight control systems.

Dr. Andeen has been awarded a patent for an automatic pilot turn coordination system. He presented a paper on nonlinear sampled-data control systems at the first international meeting of the International Federation of Automatic Control in Moscow in 1960.

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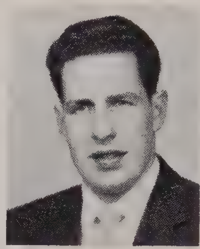

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E. Bruce Lee was born in Brainerd, Minn., on February 1, 1932. He attended Brainerd Junior College, and received the B.S. degree in mechanical engineering in 1955 and the M.S. degree in 1956 from the University of North Dakota, Grand Forks. He received the Ph.D. degree at the University of Minnesota, Minneapolis in 1960.

He joined the faculty of the University of North Dakota in 1955 as an instructor and in 1955, joined the Minneapolis-Honeywell Regulator Co., Minneapolis, as an evaluation engineer. In 1956, he transferred to the Honeywell Research Department, where he has been active in the evaluation of new concepts in automatic control, and where he is presently a senior research scientist. He has also served as a part-time instructor at the University of Minnesota, conducting courses and seminars on the theory and application of modern control engineering.

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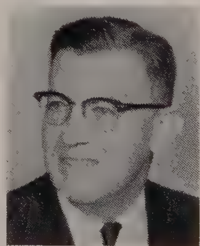

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During 1955, he was a Fulbright Lecturer and organized laboratories in automatic control systems at the University of Sydney and the University of New South Wales in Sydney, Australia.

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In 1958 he received the M.S. degree in engineering from the University of California at Los Angeles. From 1959 to the present he has been with Fairchild Astrionics Division, Wyandanch, N. Y., currently serving in the capacity of technical assistant to the technical director responsible for all divisional technical proposal activity.

Mr. Newman is a member of the Research Society of America, the American Rocket Society, and the Institute of Aeronautical Sciences.

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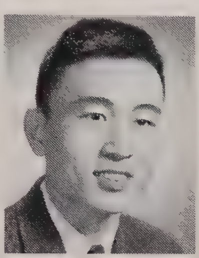

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Dr. Torng is a member of Sigma Xi.

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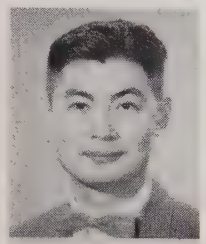
Julius T. Tou (A'53–SM'56) was born in China on August 15, 1926. He received the B.S. degree from the National Chiao-Tung University, Shanghai, China, in 1947, the M.S. degree from Harvard University, Cambridge, Mass., in 1950, and the D.Eng. degree from Yale University, New Haven, Conn., in 1952. At Harvard he was a Gordon McKay Scholar.



J. T. Tou

Following his graduation from Yale, he joined the Research Division of Philco Corporation, Philadelphia, Pa., as a project engineer. During 1955 to 1957, he was with the Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, as an assistant professor. At the Moore School, he taught graduate-level courses on feedback control theory and started a new course on digital and sampled-data control systems. In September, 1957, he joined the faculty of Purdue University, Lafayette, Ind., as an associate professor of electrical engineering. He has been a consultant to Philco Corporation and other industrial firms in the field of control systems and computers, and is the author of a comprehensive, graduate-level text, "Digital and Sampled-Data Control Systems," published by McGraw-Hill Book Co., Inc., New York, N. Y., in 1959.

Dr. Tou is a member of the AIEE, ASEE and Sigma Xi.

Notice

INCREASE IN GROUP MEMBERSHIP FEE

The PGAC is endeavoring to provide the greatest possible service to its members in the publishing of control papers. The rate of submission of good TRANSACTIONS papers continues to increase and we are nearing our goal of a quarterly TRANSACTIONS on a regular schedule. In addition, the PGAC plans to continue the practice started in 1959 of publishing the PGAC portions of the IRE INTERNATIONAL CONVENTION and WESCON RECORDS as special TRANSACTIONS issues, and will also publish IRE papers presented at the annual Joint Automatic Control Conferences. Needless to say, publication of the PGAC TRANSACTIONS costs money, and the PGAC must keep a balanced budget.

Each of the Professional Groups has its own funds, derived in part from an IRE subsidy. For a number of years the IRE provided support to the Professional Groups by matching membership fees, but this was recently changed in a manner to emphasize and promote Group publications. Thus, in 1959 the IRE supported the Professional Groups by providing one-third of all costs directly attributable to the publication of TRANSACTIONS and Newsletters. However, the rapid increase in publications (and therefore in IRE subsidies) planned by the 28 Professional Groups has forced the IRE Board of Directors to depart from strict application of the one third rule in 1960, and to place ceilings on the support provided each group. Thus, the IRE publications subsidy to the PGAC in 1960 will be limited to $3000, rather than the figure of $4000 submitted in the PGAC budget.

With the reduced level of IRE support, the PGAC balance sheet shows a negative derivative which obviously cannot continue for very long. Since it is not anticipated that the level of IRE support will be significantly increased in subsequent years, some action was clearly required. The PGAC Administrative Committee discussed this situation at its meeting in March, 1960, and decided to raise the PGAC membership fee from $2.00 to $3.00, effective July 1, 1960. The only alternative, the reduction of PGAC publications, was not considered desirable. It is noted that of the 28 IRE Professional Groups, 11 already have a $3.00 fee, and 2 have a $4.00 fee.

It is hoped that this decision of the PGAC Administrative Committee to continue present publication policies will meet with the approval of the PGAC membership.

JOHN E. WARD, *Chairman, PGAC*

Roster
of
Professional Group
on
Automatic Control

ROSTER OF PGAC MEMBERS

Listed by IRE Regions and Sections as of August 26, 1960.

REGION I

Binghamton

Barlow, J. P.
Bosman, E. H.
Dearden, Z. T.
Hudson, F. J., Jr.
Johnson, G. W.
Kilmer, F. G.
Kornbluh, A. J.
Kovalchick, Nicholas
Lazarus, Peter
Long, R. E.
Markarian, Hagop
Melville, R. W.
Miglore, Carmine
Morgan, W. A.
Northrup, R. M.
Nuss, H. C. E.
Pratt, R. L.
Richter, K. E.
Shatz, J. R.
Sitterlee, L. J.
Stankowich, J. A.
Tobin, B. M.
Wilkinson, R. H.
Wilson, W. P.
Wilts, J. R.

Boston

Abell, G. R.
Ackerman, Sumner
Adams, H. E.
Adams, P. L.
Alexander, C. S., Jr.
Anderegg, J. S.
Apsokardos, E.
Bach, R. E., Jr.
Baker, H. W.
Baker, T. S.
Ballantine, J. H.
Barker, H. J.
Barry, J. G.
Batchelder, Laurence
Beauregard, L. P.
Beecher, A. E.
Benkley, F. G.
Bennett, H. W.
Bennett, R. K.
Biernson, G. A.
Blanchard, R. L.
Bliss, Z. R.
Bodendorf, B. B.
Bosch, F. M.
Bradley, C. L.
Brew, R. A.
Brindley, W. E.
Brown, G. S.
Bullard, A. H., Jr.
Burwen, R. S.
Byck, D. M.
Capen, E. B.
Cathou, Pierre-Yves
Chandler, C. H.
Chi, Chang Hwi
Chin, B. M.
Claflin, R. E., Jr.
Clapp, C. W.
Clark, G. L.
Clements, D. F.
Collins, J. A.
Connelly, M. E.
Cowling, C. F.
Craven, R. B.
Crosthwait, J. L.
Curran, G. B.
Curtiss, W. P.
Dandreta, William
Davis, M. C.
Denner, J. A., Jr.
Dickson, A. W.
Dilg, G. C.
Dinman, S. B.
Dixon, L. H., Jr.
Djorup, R. S.
Dratch, J. E.
Duda, R. O.
Dutcher, B. C.
Dworshak, F. G.
Eachus, J. J.
Earsy, R. M.
Eckels, A. R.
Edelstein, M. M.
Edwards, L. C.
Elkind, J. I.
Engman, G. E.
Eppler, W. G., Jr.
Epstein, S. L.
Erb, D. R.
Esthimer, W. F.
Fain, Gilbert
Fallows, E. M.
Farkas, A. I.
Fernald, O. H.
Fertig, Kenneth
Fiddes, G. B.
Fitzmorris, M. J., Jr.
Forster, H. D., Jr.
Frank, W. I.
Frederick, D. K.
Freimer, M. L.
Freudberg, R. L.
Fujimoto, G. M.
Fulks, R. G.
Galagan, Steven
Gansler, J. S.
Garber, T. A.
Gelb, Arthur
Gibbs, M. M.
Gicca, F. A.
Glick, A. L.
Goldberg, David
Gordon, B. M.
Gould, L. A.
Goulder, M. E.
Graham, E. A., Jr.
Greene, A. H.
Groginsky, H. L.
Grondstra, J. W.
Gross, Michael
Grosse, P. C.
Grossimon, H. P.
Grumman, G. S.
Hagan, T. G.
Hancock, J. D.
Hanks, E. C., Jr.
Hansen, J. T.
Harley, R. W.
Harmer, J. D.
Hassan, A. R.
Hazeltine, Barrett
Hersh, A. I.
Heuchling, T. P.
Hill, R. F.
Hills, F. B.
Hills, W. L.
Ho, Yu-Chi
Hopkins, A. L.
Howard, R. A.
Huang, J. S. T.
Hui, P. S. P.
Huibonhoa, Roger
Humphrey, W. M.
Iffland, J. J.
Jaffe, R. C.
Jamgochian, Edward
Janos, W. A.
Johnson, E. P., Jr.
Kailath, Thomas
Kain, R. Y.
Kapadia, B. R.
Keith, G. E.
Kelleher, J. J.
Kerk, C. T.
King, C. M.
Kipiniak, Walerian
Klarman, K. J.
Kleinrock, Leonard
Knight, Geoffrey, Jr.
Kodis, R. D.
Kolker, R. A.
Korn, S. J.
Kramer, Robert
Krasny, L. M.
Krawiec, J. P.
Kriegsman, B. A.
Krishnayya, J. G.
Lazuk, A. V.
Leavy, P. M., Jr.
Lechner, R. J.
Lees, Sidney
Leonard, C. E.
Leonard, R. R.
Levien, R. E.
Levin, M. J.
Levy, L. S.
Lincoln, A. J.
Lloyd, D. D.
Loeffler, W. J., Jr.
Lynch, E. E.
Mahoney, T. F.
Mann, J. R.
Mark, R. B.
Martin, F. H.
Martin, L. H.
Max, S. M.
Mayer, I. B.
McCarthy, J. F.
McGinn, J. F.
McLaughlin, R. J.
McManus, R. O.
Melanson, F. J.
Mercer, W. R.
Merchant, John
Misek, V. A.
Molloy, K. H.
Morgenstern, J. C.
Mori, Hideo
Murano, Lodovico
Mundo, C. J., Jr.
Nagy, Ferenc, Jr.
Narendra, K. S.
Naylor, T. K.
Neidorf, Edward
Newton, G. C., Jr.
Nickles, J. E.
Nielsen, C. E., Jr.
Nitzsche, John
O'Brien, D. G.
Olsen, K. H.
Olsson, E. A.
Orenberg, Arthur
Ortolano, A. J.
Osburn, P. V.
Page, H. M.
Palmer, P. J.
Parke, N. G., III
Parkinson, B. N.
Passera, A. L.
Pastan, H. L.
Paynter, H. M.
Pease, W. M.
Petiteruti, A. J.
Pinckney, R. P.
Platt, H. J.
Plourde, H. S.
Potts, D. L.
Pouliot, L. J.
Preston, J. L.
Pugh, A. L., III
Pughe, E. W., Jr.
Ramamoorthy, C. V.
Reeves, R. L.
Reynolds, R. O.
Richman, P. L.
Rigby, Sherman
Rittenburg, S. E.
Rodriguez, J. E.
Rojak, F. A.
Rounds, W. A.
Rousseau, W. F.
Rowell, W. G.
Roy, A. G., Jr.
Sabin, E. A.
Saunders, N. B.
Scanlon, W. C.
Schnall, Emanuel
Schochet, J. R.
Schramm, M. W., Jr.
Schroeder, R. L.
Schwerdt, K. R.
Seifert, W. W.
Sevian, E. A.
Shappy, R. A.
Sheingold, D. H.
Shortell, A. V., Jr.
Sinclair, D. B.
Smith, B. K.
Smith, J. E.
Smyth, R. R.
Spergel, Philip
Spitzer, Edward
Stanfield, R. B.
Stata, R. S.
Stevens, J. W.
Sudduth, W. B.
Susskind, A. K.
Sutro, L. L.
Taylor, H. P.
Teager, H. M.
Teixeira, N. A.
Thoresen, J. C.
Treiber, Robert
Tsao, C. H. K.
Tunnicliffe, W. W.
Vacca, R. H.
Valpey, R. S., Jr.
Van Horn, E. C., Jr.
Van Trees, H. L., Jr.
Vanderburgh, Alexander, Jr.
Vander Velde, W. E.
Vreeland, J. B.
Ward, J. E.
Webber, R. F.
Wefald, K. O.
Weinstein, I. J.
Weiss, Thomas
Wellock, R. R.
Whitcraft, W. A., Jr.
White, D. J.
Whittaker, H. F., Jr.
Wilcox, R. B.
Wilkie, L. E.
Williams, J. J.
Williams, S. B.
Willis, J. T.
Wolf, S. H.
Wolfe, Russell
Woodruff, T. E.
Woodward, J. H.
Wyckoff, Stanley
Young, F. M.
Zieman, H. E.

Buffalo-Niagara

Archibald, W. R.
Beneke, Jack
Falk, Thomas
Gavin, R. F.
Halsted, G. P.
Hayman, R. A.
Herold, C. D.
Jones, D. R.
Lightburn, R. C.
Martin, F. C.
Meister, L. E.
Merle, C. W.
Michaelis, T. D.
Powell, F. D.
Radvany, I. E.
Rynaski, E. G.
Schneeberger, R. F.
Schultz, W. C.
Stieber, Alexander
Vaitkevicius, L. A.
Veath, M. S.

Connecticut

Bourret, C. J.
Brockett, R. I.
Brooks, F. A., Jr.
Brownell, R. M.
Capraro, R. R.
Chung, Kwangho
Corvari, John
Couturier, R. A.
Cox, D. B., Jr.
Crane, B. A.
Croft, G. T.
Dellenbaugh, F. S.
Elbe, J. H.
Ferre, M. C.
Flynn, T. F.
Georgi, E. A., Jr.
Gilchrist, E. S.
Gray, N. L.
Gutterman, A. Z.
Haas, V. B., Jr.
Hall, B. A.
Hawkins, P. J.
Helterline, L. L., Jr.
Hemstreet, H. S.
Hubbard, R. G., Jr.
Hufnagel, R. E.
Kay, Martin
Kemble, T. H.
Kidwell, N. E.
Kim, Myunghwan
Kocsis, Louis
Kowalski, H. J.
Leuze, W. F.
Lucal, H. M.
Mahoney, William
Mahoney, W. J.
Malmleaf, W. R.
Maltais, W. E.
Mapes, T. J.
Miller, R. A.
Muller, G. W.
Neill, G. W.
Nejame, Samuel, Jr.
Nelson, G. E.
Nofrey, L. C.
Northrop, R. B.
Norton, C. A.
Palmero, Albert
Penney, A. W.
Perkins, W. W.
Pichard, L. A.
Plehaty, S. L.
Poland, W. L.
Rabinowitz, C. M.
Rand, P. S.
Raposa, F. L.
Reynolds, A. C., Jr.
Rhyins, R. W.
Sontheimer, C. G.
Stasiowski, C. M.
Stephanz, G. H.
Stoker, W. C.
Swanson, J. R.
Torrance, J. H.
Turck, D. S.
Turtle, Q. C.
Tuteur, F. B.
Tutty, J. E.
Van Train, J. L.
Watson, J. M.
Willis, P. A.
Wischnia, H. F.
Yamron, Joseph
Zukowsky, W. S.
Zweig, Felix

Elmira-Corning

MacDonell, P. A.
Trementozzi, F. J.

Erie

Gray, R. B.
Rochin, D. W.

Ithaca

Digumarti, V. V. R.
Ho, Kwai Wah
Kovatch, George
Lind, E. R.
Mayer, H. F.
Phillips, T. A.
Quinlivan, R. P.

Rochester

Agalides, Eugene
Berch, W. H.
Brown, G. A.
Ceranowicz, H. N.
Chesna, John
Cotter, W. F.
Dietz, J. H.
Enslein, Kurt
Formicola, A. F.
Heit, J. C.
Kanal, L. N.
Krawitz, Max
Meserve, W. E.
Miller, H. A.
Morse, J. E.
Morris, I. A.
Pagerey, P. F.
Rodatus, E. J.
Rogers, R. J.
Sanderson, R. L.
Shepard, W. H.
Smith, G. H.
Tartanian, C. N.
Thalner, R. R.
Trost, H. K.
Trott, Marvin
Wheeless, L. L., Jr.
Williams, R. J.
Wolf, A. A.
Ye, J. W. F.

Rome-Utica

Baldridge, B. H.
Close, J. N.
Dover, B. M.
Glaser, G. J.
Isaac, M. G.
Jaros, W. F., Jr.
Rasmussen, C. P.
Ross, P. C.
Rubenstein, S. E.
Smith, D. R.
Walker, R. K.
Wendt, R. F.
Wornach, Russell
Yoder, D. F.

Schenectady

Acton, E. S.
Alexander, N. W.
Borner, E. F.
Buchhold, T. A.
Chestnut, Harold
Dabul, Amadeo
De Russo, P. M.
Dodson, G. C., Jr.
Ellert, F. J.
Kirchmayer, L. K.
Lippitt, D. L.
Pester, R. F.
Rothe, F. S.
Shuey, R. L.
Stutt, C. A.
Swann, D. A.
Troutman, P. H.
Van Den Biggelaar, Hans
Vunk, A. C.
Willman, E. S.

Syracuse

Alimansky, Mark
Altes, S. K.

Ashcroft, D. L.
Bessette, D. U.
Brady, D. J.
Brule, J. D.
Buchta, J. C.
Burke, T. H.
Clark, P. B.
Colburn, J. H.
Cottle, D. W.
Cover, N. W.
Cox, T. M. E.
Edwards, K. A.
Hannah, W. M.
Hatfield, J. P.
Johnson, G. L.
Mayo, B. R.
Mc Carthy, J. J.
Platts, G. F.
Russell, J. B., Jr.
Sproul, S. S.
Taylor, J. M.
Wallman, E. J., Jr.

Western Massachusetts

Glista, E. M.
Grooms, W. G.
Herchenreder, Herbert
Luoma, R. A.
Newsham, C. D.
Peterson, G. R.
Stang, J. L.
Van Patten, R. A.
Wright, W. D.

REGION 2

Long Island

Adams, R. R.
Adelman, Stephen
Adise, H. H.
Agree, Irvin
Amato, J. A.
Amberger, D. J.
Arguello, R. J.
Bargeski, A. J.
Behn, E. R.
Bellantoni, Juan
Boorstyn, R. R.
Boyce, J. J.
Boyer, J. H.
Braun, Ludwig, Jr.
Buehrle, W. E., Jr.
Burgess, E. G., Jr.
Burr, R. P.
Cabeen, S. K.
Cady, R. T.
Calpena, John
Cap, S. T.
Caruthers, F. P.
Chapman, P. W.
Conte, J. A.
Coren, I. B.
Costa, T. A.
Croly, J. W.
Crosby, M. G.
Cullen, W. P.
Dean, C. E.
De Gennaro, Raymond
Degnan, B. T.
De Rocher, W. L.
Detwiler, H. P.
Devieux, Carrie
Di Toro, M. J.
Dmystrasz, J. N.
Doersam, C. H., Jr.
Donop, D. A.
Dorato, P. E.
Dyer, J. N.
Edelmann, Richard
Eicher, H. G.
Eklund, K. E.
Emilio, G. A.
Eng. D. Y.
Erenburg, S. A.
Farber, Bernard
Feay, D. I.
Fertik, H. A.
Fink, Albert
Firth, L. G., Jr.
Flynn, W. J.
Fox, J. L.
Franceschini, J. B.
Frank, F. P. E.
Freeman, Herbert
Freud, E. L.
Fried, George
Friedman, E. D.
Friedensohn, George
Friedlander, M. O.
Frisch, Ivan
Geiss, G. R.
Gilbert, R. C.
Gilmore, J. F.
Gilmore, M. A.
Glick, S. M.
Glickman, Herbert
Glixon, H. R.
Gneses, M. I.
Gomes, N. A.
Goodstein, Julian
Gordon, Fredric
Gordon, R. L.
Gorelick, George
Grabbe, Dimitry
Gretz, R. W.
Griffen, K. A.
Gross, S. H.
Hall, R. L.
Hanft, C. J.
Hansen, H. R.
Harrison, Seymour
Hauerstock, Conrad
Hausmann, H. C.
Hawkes, T. A.
Hayes, N. M.
Heacock, W. J., Jr.
Herman, Sidney
Hershkowitz, Philip
Herskowitz, G. J.
Hirschel, Joel
Huber, W. J.
Jacobson, Jesse
Jaworowski, B. R.
Johl, M. J.
Joline, E. S.
Joseph, R. C.
Kalpaxis, J. A.
Katell, Emanuel
Kerner, Theodore
King, L. H.
Kinney, D. D.
Kintner, P. M.
Kirby, M. J.
Klein, R. C.
Kmiecik, J. E.
Knocklein, H. P.
Koehler, K. G.
Kohn, David
Kolakowski, K. P.
Kopp, R. E.
Kudo, Yoshito
Kuhn, K. H.
Kupferberg, Kenneth
Laspina, C. A.
Lemanczyk, J. C.
Lenefsky, Selig
Levenstein, Harold
Levinson, Emanuel
Lew, James
Lipsky, A. H.
Lipton, A. H.
Loeffler, W. J., Jr.
London, F. H.
Marcinkowski, E. R.
Marino, Joseph
Marro, M. J.
Match. M. J.
Meirowitz, R. L.
Mendel, J. M.
Meyer, D. P.
Mooney, V. J.
Moskowitz, I. W.
Mujica, T. H.
Murphy, R. B.
Nelson, A. G.
Newman, D. B.
Nykolyn, Andrew
Orgass, Richard
Orford, R. J.
Packer, Leroy
Padovano, R. J.
Parkins, George
Pearsall, G. H., Jr.
Perliss, R. E.
Peterson, H. O.
Phagan, R. J.
Preis, C. O.
Price, David
Prichard, J. S.
Purdy, P. S.
Rearwin, R. H.
Rehberg, C. F.
Reichel, Herman
Roberts, Albert
Robertson, D. J.
Rosenthal, S. A.
Ross, B. R.
Rostky, G. H.
Rowe, M. J.
Sachleben, J. H.
Saliano, Albert
San Giovanni, Carlo, Jr.
Sant Angelo, M. A.
Sayer, George
Schimsky, David
Schindwolf, Rudolph
Schneider, R. M.
Schneiderman, William
Schroeder, K. R.
Schulkind, Donald
Schuster, G. M.
Schwartz, Albert
Schwartz, Bernard
Schwartz, I. S.
Scott, J. E.
Seliger, V. H.
Seligson, A. L.
Selnick. L. L.
Selvaggio, N. J.
Shergalis, D. J.
Shulman, Irving
Sieminski, Edward
Simonton, L. J.
Slysh, Paul
Smilowitz, S. N.
Smith, H. H.
Snyder, R. L.
Soboleski, W. P.
Soreny, E. V.
Spence, H. W.
Spiegelthal, E. S.
Stark, E. W.
Stein, B. J.
Stephenson, J. G.
Storey, Moorfield, Jr.
Taboski, A. F.
Tengelsen, W. E.
Thomson, H. C.
Tiernan, F. V.
Tierney, T. J.
Tschudi, E. W.
Tsung, Pai-An
Van Dercreek, L. M.
Vinarub, M. E.
Vogel, Erwin
Waldrop, J. E., Jr.
Walker, A. C.
Walters, J. L.
Warren, S. D.
Wathen, R. L.
Weiss, D. W.
Weiss, Gerald
Werst, M. C.
Westover, T. A.
Wexler, Harold
Wheeler, H. A.
Wiesner, Leo
Wild, J. J.
Winzemer, A. M.
Witmer, F. S.
Wojcik, Edward
Wurman, Gustave
Young, D. B.
Young, V. J.
Zadoff, S. A.
Zetkov, G. A.
Zirin, M. J.

New York

Alexandro, F. J., Jr.
Annable, W. G.
Arcand, A. T.
Archbald, R. W.
Armstrong, R. W.
Arrow, Arthur
Axelrod, Jack
Banner, Leonard
Bastien, R. C.
Batchelor, J. C.
Baum, M. C.
Baumann, D. A.
Bayliss, A. L.
Beattie, J. W.
Beausalerl, W. F.
Belluck, R. E.
Bernstein, Ralph
Bickel, H. J.
Bigelow, S. C.
Bing, Charles, Jr.
Blecher, Franklin
Boggio, M. R.
Brailey, M. L.
Bromberg, N. S.
Brookner, Eli
Burfeindt, K. L.
Caplan, David
Cattelona, J. A.
Chang, S. S. L.
Chu, Henry
Cinelli, H. L.
Clemens, G. J.
Clynes, M. E.
Cohn, M. R.
Coleman, S. F.
Comninos, Robert
Connelly, J. J.
Cornetz, Walter
Cumming, M. J.
Damerell, J. B.
Defloria, R. N.
De Franco, J. J.
De Lessio, Noel
Diamond, J. E.
Dickens, S. S.
Diebold, J. T.
Di Santi, Nicholas
Di Stefano, John
Dolkas, Constantine
Drossman, M. M.
Duffy, J. J.
Eckenfelder, R. C.
Edwards, Andrew, Jr.
Egnuss, E. M.
Ellis, P. H.
Faup, J. J.
Federman, A. I.
Feinerman, Bernard
Feldman, Sidney
Fenster, Paul
Ferdman, S. C.
Fifer, Stanley
Fincke, W. H.
Fine, C. R.
Finke, H. A.
Fitzpatrick, J. J.
Fleischer, P. E.
Fleischer, Victor
Freudenberg, Boris
Friedland, Bernard
Friedman, Charles
Gabriel, E. Z.
Galloway, F. M.
Garson, E. L.
Gatzke, F. W.
Gilman, G. W.
Gindoff, Martin
Glantz, L. M.
Glasser, L. J.
Golden, Donald
Goldman, G. M.
Goldsmith, A. N.
Goldstein, Harvey
Goldstein, Harold
Goodman, P. H.
Gorczycki, Edmund
Grayson, L. P.
Greenberg, E. L.
Gronner, Alfred
Gross, R. I.
Guenther, Richard
Haddad, R. A.
Hannibal, J. O.
Harrison, G. S.
Harty, L. T.
Helbig, W. L.
Heller, Samuel
Hellerman, Herbert
Hillman, H. D.
Hilton, A. M.
Hoekstra, Robert, Jr.
Hoffman, J. D.
Holzman, Sandor
Hough, J. B.
Houldin, R. J.
Hull, R. B.
Humphrey, R. M.
Hung, J. C.
Iddings, G. E.
Jacobsen, T. L.
Jorysz, Alfred
Kardauskas, Edmund
Kassel, Aaron
Katz, M. D.
Kellner, Richard
Kilian, L. H.
Klein, S. S.
Koch, I. I.
Koepcke, R. W.
Kranc, George
Krueger, A. M.
Ku. W. H.
Kuzmyak, M. G.
Kwo, T. T.
Lee, C. T.
Lee, Song-Pai
Legendziewicz, J. S.
Letzter, A. O.
Levy, S. M.
Liebman, P. M.
Lindner, N. I.
Lotito, L. A.
Lovett, R. S.
Lowdenslager, J. R.
Machorro, R. A.
Mahar, T. F.
Mahony, D. J.
Mahrous, Haroun
Mark, Marvin
Markham, G. W.
Martin, R. R.
Martinson, F. L.
Massell, Edward
McPherson, D. L., III
Meindl, J. D.
Messinger, M. P.
Molinelli, R. J.
Mon, W. C. W.
Morrow, S. R.
Murphy, W. J.
Nadler, W. L.
Napolitano, M. R.
Narici, L. R.
Narud, J. A.
Nelson, N. G.
Newman, Leonard
Novick, William
Olsen, E. C.
Oppenheimer, H. N.
Oswald, Henry
Parr, A. F. W.
Parvulescu, Antares
Paschetto, E. J.
Pearson, H. A.
Pedersen, C. R.
Pedowicz, J. M.
Perlmutter, Al
Perry, Lawrence
Porter, R. W.
Poss, Eliasz
Preiss, R. J.
Purdy, R. L.
Quan, Stanley
Ragazzini, J. R.
Raship, Manush
Rattner, Arnold
Reeves, J. F.
Reiss, R. A.
Reynolds, Gibson
Rider, P. M.
Riley, W. B.
Roberts, R. P.
Roehrig, H. W.
Roman, Saul
Rosen, Robert
Rosenberg, H. J.
Rosenthal, Lee
Roth, J. E., Jr.
Rothenstein, Stuart
Royer, G. H.
Rudofsky, B. I.
Rugge, R. A.
Sarachik, P. E.
Schoenfeld, R. L.
Schwartz, J. H.
Schwarz, R. J.
Semer, Mortimer
Shalev, Yosef
Shapiro, Eugene
Shapiro, Paul
Sherman, Seymour
Shinners, S. M.
Shooman, M. L.
Shostack, Sheldon
Silberberg, M. Y.
Sinclaire, F. S.
Sippel, R. J.
Slavin, M. J.
Slevin, R. F.
Stalemark, F. R. V.
Steinman, G. R.
Sterman, I. I.
Stern, Kurt
Suarez, Anthony
Sussman, Nathan
Swain, J. A.
Swartz, Jerome
Szabo, J. A.
Thomas, R. O.
Truxal, J. G.
Tsilibes, G. N.
Tucker, A. G.
Turchiano, M. W.
Turczyn, W. A.
Turetsky, Norman
Utsunomiya, Toshio
Vigants, Arvids
Vogeler, R. A.
Voulgaris, N. C.
Walton, J. S. V.
Watkins, J. E.
Weidberg, E. H.
Weiner, George
Weintraub, Irving
Weisman, Irving
Weiss, A. J.
Welt, Sanford
Wernick, J. I.
Wertheim, L. A.
Willett, Edward
Winkelstein, R. A.
Wolfson, H. S.
Wolfson, Richard
Wright, Donald
Young, Kwok-Ping
Young, R. O.
Zourides, V. G.

Northern New Jersey

Anderson, N. E.
Antonazzi, F. J.
Aubrey, Gerald
Bahls, W. E.
Bailey, E. M.
Bearman, A. L.
Beeny, L. S.
Berenyi, Szilard
Bertuccio, T. D.
Bethke, J. R.
Blauvelt, D. H.
Bodnar, E. R.
Bogner, Irving
Boubli, E. J.
Bowker, M. W.
Brinkman, J. D.
Brown, B. D.
Bucklin, K. G.
Byrne, E. R.
Cowles, W. W.
Crump, E. E.
Cynamon, J. J.
Darbie, A. M.
Dieperveen, Neal
Digrindakis, Michael
Di Nardo, R. P.
Doba, Stephen, Jr.
Donaldson, W. G.
Doniger, Jerry
Drenick, Rudolf
Ebersol, E. T., Jr.
Elwell, H. G., Jr.
Erbrecht, G. F.
Fineman, T. P.
Finkelstein, David
Foster, R. M.
Fristensky, C. A.
Fromer, Morton
Gaddner, J. L.
Glasberg, Martin
Gordon, M. J.
Grandmont, P. E.
Griebell, C. A.
Grossman, A. J.
Helgeson, B. P.
Helms, H. D.
Hinger, F. J.
Hires, J. B.
Kahant, C. G.
Kayser, J. F.
Kaszerman, Philip
Kierce, Eugene
Kinariwala, B. K.
Kiss, M. L.
Kitsopoulos, S. C.
Krell, Joakim
Kuehler, H. R.
Kunkel, E. A., Jr.

Lapham, C. J.
Lazos, N. J.
Leed, Daniel
Leeds, Myron
Lieberman, I. J.
Litman, L. S.
Louis, R. A.
Lozier, J. C.
Lucas, R. M.
Lunney, R. E.
Mandel, Mark
Mathews, M. V.
Mayo, J. S.
McAllister, N. F.
McCrory, J. R.
McDaniel, W. J.
McIntyre, J. W.
McMurtrie, C. L.
Meinholtz, H. J.
Melville, J. R.
Mindes, A. S.
Moskowitz, Carl
Mount, Ellis
Mueller, P. L.
Mulligan, J. H., Jr.
Murray, A. A.
Nash, R. A., Jr.
Nelson, W. L.
Newhall, E. E.
Nielsen, D. M.
O'Donohue, J. P.
Ossanna, J. F., Jr.
Otis, A. N., Jr.
Painter, J. G.
Panter, P. F.
Paradise, R. Y.
Perlis, H. J.
Peters, D. P.
Peters, G. J.
Pfersch, G. H., Jr.
Podell, R. L.
Potash, Jerome
Pravetz, R. M.
Reilly. R. A.
Richards, G. P.
Roberge, J. K.
Robinson, A. S.
Rosenthal, C. W.
Rossi, Stephen
Russell, F. A.
Rutherford, E. C.
Sadvary, J. W.
Sapiro, A. M.
Savi, Lembit
Schmal, R. L.
Schneider, W. A.
Seergy, C. M.
Shangraw, C. C.
Shapiro, Oscar
Skwarek, F. J.
Slana, M. F.
Sliwa, R. A.
Smith, E. J.
Sobol, L. N.
Sokol, Stanley
Solomon, R. H.
Staluppi, J. P.
Stickle, R. L.
Stiefel, K. E.
Sutton, L. E., III
Szentirmai, George
Thompson, C. F.
Thompson, T. H.
Torrey, L. W., Jr.
Thomas, D. E.
Uebelhoer, H. K.
Unger, Arnold
Veazie, Waldemar
Vinkman, V. V.
Walker, H. R.
Wall, R. D.
Warren, J. E.
Wetzel, K. C.
Wilde, A. E., Jr.
Woodman, R. A., Jr.
Zemanian, A. H.

Princeton

Amarel, Saul
Basson, D. E.
Beck, G. A.
Berkeyheiser, J. E., Jr.
Crooke, A. W.
De Versterre, W. I.
Downie, D. E.
Drath, D. G.
Garretson, E. B.
Graber, G. F.
Grasselli, Antonio
Grissler, H. W.
Gross, Josef
Hoedemaker, R. W.
Kang, Chi Lung
Kaplan, K. R.
Kerr, R. B.
Knapp, J. Z.
Kulvin, R. D.
Lannary, John
Lichtblau, G. J.
Lyon, C. C.
Maitra, K. K.
Mitchell, G. J.
Norton, J. A.
Rogers, A. E.
Ruble, G. B.
Schofield, C. R.
Schumacher, R. B. F.
Sklansky, Jack
Smith, G. W., Jr.
Surber, W. H., Jr.
Truitt, T. D.
Wendt, F. S.
Westneat, W. S., Jr.
Wolin, Samuel

REGION 3

Atlanta

Bradfute, G. A., Jr.
Chambers, J. W., Jr.
Davidson, R. C.
Dupree, J. E.
Eckel, J. R., Jr.
Feaster, W. M.
George, C. M.
Jimenez, G. L.
Johnson, D. C.
Lingar, W. E.
Robertson, D. W.
Smith, C. E.
Sparkman, R. W.
Tryon, W. W.
Ziegler, N. F.

Baltimore

Ausfresser, H. D.
Axelby, G. S.
Barrack, C. M.
Beckman, N. W.
Behm, G. T.
Bentley, F. C.
Blotzer, W. L.
Burns, J. E.
Bush, A. G., Jr.
Choksy, N. H.
Crosby, R. M.
Dick, D. N.
Eaton, T. T.
Edwards, R. L., Jr.
Fischer, P. P.
Fitzhugh, H. S., II
Gambrill, R. D.
Gegely, W. D.
Gessner, Urs
Glaser, E. M.
Glover, C. C.
Groszer, A. J., Jr.
Hart, J. J., Jr.
Horn, R. E.
Huddleston, F. J.
Hunsicker, J. E.
Hurley, W. A.
Jackson, H. L.
Jackson, J. H.
Jenkins, J. L.
Jentilet, Adam
Jones, L. G. F.
Jones, W. N.
Kalman, R. E.
Karpa, Paul
Kay, J. S.
Kegel, A. G.
Kernan, Paul
Kogan, A. J.
Lai, D. C.
Leahy, F. N.
Lee, Marshall
Longuemare, R. N., Jr.
Mabry, F. S.
McDonough, R. N.
McEntryre, D. E.
Miller, L. A.
Mortimer, T. S.
Osborne, E. F.
Penabad, Joseph, Jr.
Plath, R. H.
Pulitano, F. J.
Rosenberry, W. W.
Sabbagh, H. A.
Sluss, W. C., Jr.
Spink, P. G.
Stebbins, W. J.
Stephenson, J. O.
Taragin, Saul
Tates, Robert
Thomas, J. F.
Visher, W. A.
Wells, R. W.
Whitcomb, M. F.
Williams, A. D.

Central Florida

Ferguson, R. E.
Fletcher, R. C.
Heins, D. E.
Houston, E. E., Jr.
Pelchat, G. M.
Shirley, R. C.
Sumner, L. E.
Weissenborn, H. E.
Young, W. L.

Florida West Coast

Adkisson, W. M.
Ainsworth, F. W.
D'Arpa, A. G.
Dingley, E. N., Jr.
Dowell, W. B., Jr.
Hoover, J. C.
Inman, T. F.
Lampkin, G. F.
Rosenzvaig, J. E.
Saraydar, R. A.
Scheidenhelm, Ralph
Shalloway, A. M.
Uglow, K. M., Jr.
Wells, E. L.

Gainesville

Elgerd, O. I.
Schenk, D. D.

Huntsville

Blanton, J. E.
Bradley, B. C.
Casey, C. W.
Church, J. B., Jr.
Cornutt, H. D.
Farmer, R. H.
Giltinan, T. L.
Kulberg, L. E.
Maddox, W. V.
Otto, W. F.
Salonimer, D. J.
Schumann, Fred
Seashore, C. R.
Stifler, W. W., Jr.

Miami

Cook, H. A.
Lahue, P. M.
Robinson, D. A.
Rose, R. M.
Rosencranz, Robert
Witt, A. J. B.

North Carolina

Littleton, W. W.

North West Florida

Clifton, C. L.
Gould, E. E.
Smith, J. R.

Orlando

Alexander, J. E.
Bendett, R. M.
Brown, J. O.
Buchan, J. F.
Day, W. R.
Duval, A. N.
Gray, A. R.
Grider, K. V.
Karres, G. E.
Lanzkron, R. W.
Le Gare, J. M.
Murray, J. S.
Nichols, R. C.
Painter, Parker, Jr.
Rhodes, D. R.
Schindelin, J. W.
Schwab, W. G.
Spoon, G. E.
Spore, H. B.
Stimmel, G. L., III
Stringer, K. E.

Philadelphia

Adler, E. A., Jr.
Affel, H. A., Jr.
Aires, R. H.
Alexander, F. C., Jr.
Alperin, N. N.
Hillcrest, J. M.
Anderson, W. G.
Astheimer, Walter
Auerbach, I. L.
Bachofer, H. L.
Barger, J. R.
Bechert, T. E.
Beck, M. R.
Beck, Cyrus
Berkowitz, R. S.
Bernstein, Fred
Beter, R. H.
Bikle, A. S.
Blasberg, L. A.
Bolgiano, D. R.
Boonin, L. I.
Boreen, H. I.
Brucklacher, J. E., Jr.
Busby, J. W.
Buxton, P. T.
Bycer, B. B.
Campanella, M. J.
Capron, R. W.
Cecal, J. A.
Ching, S. W. F.
Chronister, W. M.
Chudleigh, W. H., Jr.
Clutz, D. A.
Colehower, C. H., Jr.
Collins, D. H.
Connaught, P. M.
Cooley, C. C., Jr.
Cox, R. L.
Crowhurst, W. R.
Curtin, W. A.
Dahlberg, R. S., Jr.
Dahlin, E. B.
D'Andrea, L. L.
Davis, R. W., Jr.
Deacon, N. E.
Desautels, G. L.
Dewey, C. P., Jr.
Dickens, B. L.
Dunkel, H. D.
Durfee, G. H.
Fabioli, L. F.
Fairman, F. W.
Fath, J. P.
Faust, A. C.
Faustini, Carlo
Fegley, K. A.
Foley, G. M.
Friend, A. W.
Frosch, J. A.
Fuchs, A. M.
Fujimoto, Akira
Fuller, L. C., Jr.
Gale, Morten
Gardiner, F. J.
Gluck, S. F.
Goff, K. W.
Gottschalk, J. M.
Greely, D. K.
Gregory, T. R.
Grim, E. D.
Hartnett, E. J.
Hattori, Yoshio
Hellerman, Herbert
Herrmann, J. E.
Hitt, J. J.
Hoover, E. W.
Horton, R. L.
Houpt, G. K.
Howson, R. D.
Hsieh, P. K.
Hughes, C. J.
Hughes, P. F.
Huyett, W. I.
Iadicicco, R. A.
Isom, W. R.
Jacaki, W. B.
Jackson, A. S.
Johnson, M. C.
Johnston, Winfield
Kashmar, E. J.
Kasowski, S. E.
Keat, J. E.
Kozak, W. S.
Kozikowski, J. L.
Krantz, F. H.
Ku Yu, H.
Lamonte, R. J.
Lannutti, A. P.
Lathrop, P. A.
La Verghetta, F. E.
Lazarus, F. F.
Lazinski, R. H.
Levy, A. S.
Lieb, A. B.
Linhardt, R. J.
Lisicky, A. J.
Lockhart, J. C.
Loev, David
Loftus, J. A.
Low, Frank
MacDonald, R. D.
Macqueene, P. H.
Madera, A. J.
Maguire, H. T.
Mao, Lai-Yen
Marks, T. P.
Marx, S. H.
Mayer, Robert
McClure, R. W.
McCracken, L. G., Jr.
McWilliams, C. R.
McWilliams, G. R.
Michener, Robert
Miller, G. F.
Moll, W. H.
Niquette, R. P.
O'Brien, J. F., Jr.
Osbahr, B. F.
Oseas, Jonathan
Paskman, Martin
Perecinic, W. S.
Pierce, O. B.
Porter, J. W.
Potosky, Maurice
Rangachar, H. V.
Rawling, A. G.
Raymond, R. E.
Reich, Alfred
Renfrow, R. J.
Richter, Filmore
Risse, J. A.
Roede, J. B.
Rogers, D. A.
Rogers, R. F.
Rose, F. L.
Roth, H. H.
Rudnick, J. J.
Rudofsky, Samuel
Sarafin, E. E.
Scheuer, Harry, Jr.
Schreiner, W. A.
Seawall, W. N.
Seliga, T. A.
Shen, D. W. C.
Shucker, Sidney
Shulman, Louis
Sink, J. A.
Slocum, L. V.
Smith, D. B.
Sobolewski, Frank
Sorkin, C. S.
Sposato, B. J.
Strip, Joseph
Stubbs, G. S.
Sun, Hun-Hsuan
Swarthe, Eric
Taenzer, Erwin
Ur, Hanoch
Voorhoeve, E. W.
Walters, W. R.
Ware, W. E.
Weisenberger, A. J.
Weiss, Eric
Wills, W. P.
Wirtz, E. L.
Wolin, Louis
Woodcock, V. E.
Yang, Tsute

South Carolina

Brooks, M. J.
Miller, A. H.
Noland, J. H., Jr.
Pippin, R. F., Jr.
Ross, J. D.

Virginia

Andrews, R. E.
Branscom, G. A.
Burrill, G. P.
Campbell, R. C.
Carley, R. R.
Cockrell, W. D.
Fadeley, J. H.
Fourdriat, E. C.
Gregory, C. A., Jr.
Harvey, G. L.
Howell, W. E.
Langley, L. W.
Pettus, W. W., IV
Pickett, R. T., III
Quarles, L. R.
Thomas, J. E. D.
Williams, M. L.
Zaluski, J. P.

Washington, D. C.

Allen, D. A.
Anderson, S. F., Jr.
Baird, T. M.
Bass, C. A.
Beckham, J. M.
Britner, R. O., Jr.
Bush, G. B.
Chu, Yaohan
Clapham, Robert
Clarke, A. S.
Connelly, E. M.
Coyle, R. J.
Davies, G. L.
Davis, T. R.
Diels, J. C.
Dillon, J. C., Jr.
Edward, C. E. H.
Enos, R. M.
Finkel, Abraham
Fleming, J. J.
Fogel, L. J.
George, S. F.
Gorozdos, R. E.
Gouge, J. R., Jr.
Hargraves, J. F.
Hauser, Herbert
Hibbard, W. D., Jr.
Hocking, L. J.
James, W. G.
Janow, Carl
Karp, M. A.
Kiehn, Gunter
Kirshner, J. M.
Lee, F. M.
Lee, J. D.
Levin, Alexander
Little, J. L.
Mallin, J. A.
McIntyre, R. T.
Mischler, E. F.
Moore, T. E.
Morgan, B. S., Jr.
Morrisey, J. A.
Nelson, M. W.
Nisson, C. J.
O'Hara, J. J., Jr.
Ostaff, W. A.
Paden, D. R.
Pai, S. M.
Partello, P. E.
Pefferman, E. B.
Penniman, I. B.
Pike, W. N.
Poor, V. D.
Price, H. W.
Prival, H. G.
Reagen, E. J.
Regardh, C. B.
Roberts, G. L., Jr.
Rogers, A. L.

Rolinski, A. J.
Rozanski, R. R. A.
Scott, D. G.
Shapiro, Gustave
Shepard, D. H.
Shimabukuro, Kazuo
Smith, W. R., III
Snow, C. E.
Stuhrenberg, M. B.
Talkin, A. I.
Trott, J. E.
Offelmsn, M. R.
Van Lunen, R. D.
Varhus, Henry
Viera, Frank, Jr.
Waterman, Peter
Watkins, P. L.
White, C. F.
Wimmer, P. L.
Witow, M. I.
Zastrow, K. D.

REGION 4

Akron

Batcher, K. E.
Bolen, N. E.
Chappell, W. M.
Colletti, Nicasio
Flowers, H. L.
Haas, D. L.
Hann, D. D.
Hermann, P. J.
Ingalls, R. S.
Lake, R. B.
Lambert, C. O.
Meilander, W. C.
Miller, J. H.
Pool, L. C.
Rogers, C. L., Jr.
Ryburn, P. W.
Snyder, E. W.
Sutherland, E. B.
Svala, C. G.
Toman, W. J. V.
Yochelson, Saul

Central Pennsylvania

Brown, J. L., Jr.
Burkey, J. R.
Chackan, H. G.
Harvey, H. B.
Herman, E. B.
Holbert, W. G.
Lawther, J. M.
Oblinger, J. T.
Seeley, R. M., Jr.
Wolfe, R. E.

Cincinnati

Berg, D. F.
Bernhard, G. J.
Boehm, J. F.
Buden, David
Burnett, L. A.
Culy, G. E.
Cunningham, G. W., Jr.
Engelmann, R. H.
Georger, L. J.
Gorker, G. E.
Hellen, R. J.
Herrin, C. B.
Hulstrand, B. E.
Knoer, Howard
Nistico, Frank
Sapp, R. S.
Westmark, J. E.
Zupansky, Milo

Cleveland

Bolz, R. W.
Costanza, J. L.
Cowan, R. A.
Davis, Samuel
Edling, E. A.
Flick, J. R.
Gooch, B. R.
Grasson, Walter, Jr.
Hart, C. E.
Hornfeck, A. J.
Jackson, Warren, Jr.
Kinkaid, J. C.
Kliever, W. H.
Klock, H. F.
Kralovic, J. J.
Madson, L. P.
Majercak, J. V.
Miller, H. J.
Miro-Nicolau, Jose
Murray, J. E.
Nadkarni, D. D.
Phillips, W. E., Jr.
Pryatel, R. D.
Saltzer, Charles
Shepherd, B. R.
Smith, C. E.
Sturman, J. C.

Columbus

Bain, T. D., Jr.
Bishop, A. B., III
Brown, R. A.
Burgener, R. C.
Butterworth, G. S.
Chope, H. R.
Cohen, Donald
Compton, R. T., Jr.
Conlon, R. J.
Cosgriff, R. L.
Coulter, N. A., Jr.
Cummins, M. M.
Ebert, D. H.
Jackson, S. P.
Kirschbaum, H. S.
Levadi, V. S.
McCaslin, C. D.
McFarland, R. S.
Mobley, M. D.
Moll, Magnus
Rose, G. C.
Sapp, E. R.
Schulman, B. L.
Schultz, R. W.
Smith, R. D.
Spell, R. W.
Tai, Chen-To
Tamplin, G. R.
Ulbrich, E. A.
Weimer, F. C.

Dayton

Bentley, R. M.
Bornhorst, K. F.
Cook, E. E.
Curtis, L. L.
Fenton, R. E.
Grimm, F. W.
Hamann, R. A.
Helf, William, Jr.
Johannes, R. P.
Kerbs, W. A.
Papaiconomou, George
Peterson, L. S.
Roig, R. W.
Sansom, F. J.
Simopoulos, N. T.
Thompson, J. P.
Wolaver, L. E.
Zachar, J. E.

Detroit

Barcus, Ronald
Bayma, R. W.
Bays, K. L.
Becher, W. D.
Benaglio, R. V.
Braun, Herman
Brown, L. R.
Bublitz, A. T.
Burr, Harold
Chaney, L. W.
Chuang, Kuei
Chute, G. M.
Cooper, K. R.
Crawford, W. N., Jr.
Da Roza, F. G.
De Muro, T. F.
Edwards, C. M.
Elrod, B. D.
Epley, D. H.
Farrah, H. R.
Fife, D. W.
Foulke, J. A.
Gaskill, R. A.
Gilbert, E. O.
Gilbert, E. G.
Harding, W. G.
Hartford, T. W.
Jania, Z. J.
Johnson, E. C., Jr.
Johnson, S. M.
Karwas, F. C.
Kazda, L. F.
Kleekamp, C. W.
Kobayashi, H. S.
Kohr, R. H.
Kotoucek, W. R.
Lindahl, C. E.
Martens, H. R.
McKelvie, J. L.
Menninga, Gysbert
Miller, F. L.
Nakagawa, Noriyika
Nauts, D. F.
Nixon, J. D.
O'Neal, R. D.
Ormsby, R. D.
Page, C. V.
Patton, H. W.
Quin, W. J.
Rane, D. S.
Rauch, L. L.
Retko, Edward
Robinson, D. F.
Robinson, G. H.
Rosenstein, M. D.
Scott, L. F.
Seleno, A. A.
Smith, D. F.
Smith, Wray
Strand, John
Sullivan, David
Taplin, L. B.
Thomas, R. L.
Tokad, Yilmaz
Ullman, H. L.
Wallace, V. L.
Webber, R. C.
Weimer, W. R.
Whiteside, A. E.

Pittsburgh

Aronson, M. H.
Bean, W. C.
Bhavnani, R. H.
Bossart, P. N.
Bright, R. L.
Byerly, R. T.
Calvert, J. F.
Canter, E. L., Jr.
Chen, Kan
Cilyo, F. F., Jr.
Coates, R. S.
De Berry, E. P.
Decker, R. O.
Eggers, C. W.
Fekete, J. M.
Ferguson, R. W.
Ford, D. J.
Gatto, J. J.
Golla, E. F.
Hyypolainen, E. K.
Kaufman, W. M.
Kreer, J. B.
Leeds, J. V., Jr.
Li, Ching-Chung
Little, D. R.
Maher, J. I.
Mathias, R. A.
Mucci, Geno
Newell, W. E.
O'Donnell, J. J.
O'Shea, R. P.
Rathbone, D. E.
Rau, F. J.
Robl, R. F., Jr.
Rogers, L. J.
Saucedo, Robert
Schindler, D. G.
Schiring, E. E.
Schwindt. A. J.
Severance, J. C.
Sippach, F. W., Jr.
Sze, Tsung Wei
Thompson, F. T.
Vogt, W. G.
Wolford, J. E.

Toledo

Ewing, D. J., Jr.
Leuck, D. D.
Murley, E. M., Jr.

Western Michigan

Alway, C. D.
Burke, R. H.
Miller, J. J., Jr.
Mort, V. A.
Rufener, R. E.
Weiss, Robert

Williamsport

Seksinsky, J. J.
Young, G. D.

REGION 5

Cedar Rapids

Berkovec, J. W.
Buroker, G. E.
Hattendorf, E. R.
Hedgcock, W. T., Jr.
Hoffman, R. E.
Keser, P. D.
Kopp, R. W.
Ljungiwe, A. L.
Schultz, E. L.
Shepard, A. R.

Chicago

Arnow, C. L.
Askew, W. J., Jr.
Auth, L. V., Jr.
Axelrod, L. R.
Bauman, P. A.
Beuligmann, R. F.
Bielenin, Andrew
Bilsens, Gunars
Blyth, R. A.
Bonk, J. T.
Borkovitz, H. S.
Boyd, D. M., Jr.
Brieske, G. F.
Brule, F. J.
Brubaugh, R. T.
Bullen, C. V.
Bulliet, L. J.
Carter, Robert
Chorney, P. L.
Coulter, W. H.
Crowther, E. A.
Cruz, J. B., Jr.
Deterville, R. J.
Dikinis, D. V.
Donalek, P. J.
Druz, W. S.
Dumay, E. C.
Egan, J. F.
Eilers, C. G.
Ekstein, D. G.
Epley, D. L.
Even, J. C., Jr.
Fernandez, D. T.
Fett, G. H.
Foster, G. E.
Frankart, Wm. F.
Fullerton, W. T.
Gascon, R. R.
Gavcus, Stanley, Jr.
Ghosh, H. N.
Glomb, J. D.
Goodrich, E. W.
Gourishankar, Vembu
Green, D. G.
Gregory, E. C.
Hannon, P. H., Jr.
Hansen, T. A.
Haugh, C. F.
Hedvig, T. I.
Hodge, Bartow
Hoffman, R. S.
Holman, W. J.
Huber, E. A.
Hutson, D. E.
Jenness, R. R.
Johnson, T. L.
Jones, E. C., Jr.
Jones, R. W.
Kabrisky, M. J.
Kafka, R. W.
Kantner, H. H.
Klem, R. F.
Kott, W. O.
Kuo, B. C. I.
Lafferty, V. C.
Laney, B. H.
Lesnick, R. N.
Lewis, H. A.
Leysath, W. Q.
Liu, Ruey-Wen
Ma, H. J.
Maginot, J. J.
Maier, R. H.
Manos, G. P.
Markunas, R. S.
Marshall, F. J.
Martin, J. W., Jr.
Melman, Myron
Meyer, Andrew
Michels, R. A.
Minarcik, G. P.
Mittelmann, Eugene
Moe, M. L.
Murphy, G. J.
Myers, J. E.
Nelson, G. L.
Norris, P. C.
Nuber, S. E.
Oldham, K. W.
Overlie, P. T.
Pallissard, J. W.
Peterson, F. C.
Pomon, J. D.
Pugsley, J. H.
Rao, S. N.
Reukauf, D. C.
Rockwood, C. C.
Sayles, H. L.
Shewan, William
Srinath, M. D.
Stehman, Carl
Steinhoff, N. K.
Suhre, M. E., Jr.
Tempka, J. A.
Thake, R. F.
Thomas, R. G.
Toepfer, R. E., Jr.
Tyler, L. A.
Van Bosse, J. G.
Van Ness, J. E.
Van Valkenburg, M. E.
Vitous, J. P.
Volk, J. E.
West, R. L.
White, E. S.
Whitmer, M. H.
Wieczorek, A. B.
Win, Soe Myint
Wishner, R. P.
Yau, Sik-Sang

Evansville-Owensboro

Jordan, J. D.

Fort Wayne

Cronkwright, R. E.
Curl, G. W.
Hallmark, C. E.
Johnson, D. L.
Kendall, P. E.
Phillips, T. H.
Richards, G. A.
Skogfeldt, H. G.
Williams, W. J., Jr.

Indianapolis

Anderson, A. L.
Chassin, C. A.
Cress, L. E.
Ekland, J. D.
Ervin, E. E.
Evans, R. A.
Eveleigh, V. W.
Geldmacher, R. C.
Gibson, J. E.
Givan, R. L.
Gregg, P. S.
Hamel, L. L.
Harrison, R. E.
Hemdal, J. F.
Joseph, P. D.
Knoop, D. E.
Landgrebe, Dave
Meditch, J. S.
Nisewander, A. L.
Ogborn, L. L.
Van Ostrand, W. F.
Raible, R. H.
Schaffner, J. S.
Skinner, G. M.
Stanley, P. E.
Tou, Julius
Vangelakos, Constantine
Whetsel, R. C.
Womack, B. F.

Louisville

Ebaugh, D. P.
Erdman, B. K.
Hill, J. C.
Morris, M. E.
Navarro, S. O.
Risk, P. M.
Schiewe, A. J.
Whitfield, J. E., Jr.

Milwaukee

Ackmann, J. J.
Bennett, T. H.
Birkemeier, W. P.
Boutelle, J. O.
Brandt, R. T.
Briedis, Gunars
Buell, C. D.
Calcai, P. P.
Fitzgerald, R. R.
Genthe, W. K.
Goudy, P. R.
Gruszynski, R. T.
Grzelak, T. A.
Hill, R. G.
Hughes, G. C.
Jensen, K. S.
Johnson, R. R.
Koehler, C. W.
Kopania, A. A.
Kriofsky, T. A.
Krueger, R. J.
Leeson, J. L.
Lemere, J. P.
Ligman, J. R.
Limpel, E. J.
Lindemann, A. W.
McGrath, R. J.
Maher, J. E., Jr.
Mertz, R. L.
Murrish, C. H.
Oberlander, L. E.
Pierce, R. L.
Pinnow, R. L.
Pozorski, J. J.
Rideout, V. C.
Sabroff, R. R.
Sackett, R. W.
Salava, R. F.
Schlicht, R. W.
Schutten, H. P.
Sherfey, M. O.
Smith, C. C.
Soli, R. H.
Thelen, E. S.
Thiele, J. F.
Trinkner, C. C.
Vander, W. H.
Wei, Yin-Min
West, B. M.
White, A. L., Jr.

Omaha-Lincoln

Bashara, N. M.
Fritz, G. L.

So. Bend-Mishawaka

Hansen, A. G., Jr.
Hoffman, C. H.
Klimek, T. F.
Nymberg R. J., Jr.
Saletta, G. F.
Warnke, T. P.

Twin Cities

Adams, G. E.
Alderson, R. C.
Bargen, D. W.
Bartlett, V. W.
Beck, V. W.
Benassi, D. A.
Butz, A. R.
Eglite, J. A.

Fjelsted, R. P.
Geronime, R. L.
Gerth, D. E.
Gise, F. G., Jr.
Gustafson, H. A.
Hardenbergh, G. A.
Harvey, P. O.
Heiser, R. K.
Hille, L. S.
Josephs, H. C.
Kershaw, J. A.
Kerske, J. F.
Ketchum, J. R.
Kiene, R. C.
Kile, F. O.
Kinnen, Edwin
Knoblauch, Arthur, Jr.
Kukuk, H. S.
Larson, P. S.
Lode, Tenny
Lundquist, P. P.
Macomber, G. R.
Markusen, D. L.
McLane, R. C.
Muckenhirn, O. W.
Nellis, W. M.
Neudecker, R. M.
Newman, James
Nordstrom, J. E.
Prom, G. J.
Reed, M. W.
Schuck, O. H.
Senstad, P. D.
Stone, N. T.
Storm, J. F.
Swanlund, G. D.
Tsui, Yau Tzong
Tu, J. C.
Windsor, R. N.

REGION 6

Beaumont-Port Arthur

Blankenship, G. E.

Dallas

Abraham, R. P.
Albertson, H. D.
Anderson, R. P.
Barnett, M. L.
Blanning, R. B.
Braginton, P. R.
Brown, K. C.
Chun, M. E.
Clinger, A. H.
Cook, C. R., Jr.
Corwin, T. L.
Day, W. J.
Dowell, K. P.
Drennan, J. B.
Dubose, G. P.
Evetts, S. G.
Faith, W. O.
Gray, C. T.
Hennen, F. M.
Hull, J. E.
Hutson, R. N.
Jones, J. F.
Koeijmans, G. D.
Lavergne, A. J., Jr.
Livingston, R. M.
Mills, R. C.
Ocnaschek, F. J.
Petrasek, A. C.
Pittman, P. D., Jr.
Redmond, W. G., Jr.
Sarrafian, G. P.
Stanton, A. N.
Syptak, D. M.
Tatum, F. W.
Teasdale, A. R., Jr.
Tschudi, D. E.
Wadel, L. B.
Weaver, S. M.
Wilhelm, E. S.
Ziemer, D. R.

Denver

Anthony, C. A.
Braunagel, M. V.
Capehart, M. E.
Case, C. W.
Davis, V. M.
Donnelly, J. B.
Finch, M. D.
Gatterer, L. E.
Harrison, F. H., Jr.
Hart, W. G., Jr.
Kurz, D. F.
Maher, R. A.
Richards, L. G.
Schneebeck, D. A.
Smith, G. W.
Stacey, D. S.
Stewart, C. H., II
Wakefield, J. P.

El Paso

Lovitt, S. A.
Rojas, A. M.
Schroeder, K. G.
Weiner, M. C.

Fort Worth

Allen, Rufus, Jr.
Blackwell, W. A.
Buehrle, C. D.
Egan, D. G.
Elliott, J. E.
Forester, J. R.
Hale, R. A.
Hines, R. L.
Jiles, C. W.
Moore, N. E.
Mowery, D. K.
Muery, R. W.
Seale, I. A.
Slagle, G. M., Jr.
Tucker, B. J.
Watkins, O. E.
Webb, V. C.
Welter, N. E.

Houston

Brook, A. H.
Cohn, Diter
Erath, L. W.
Gass, F. G.
Haill, H. K.
Hattz, R. F.
Johanson, K. A., Jr.
Johnson, R. M.
Khudrathullah, M. F.
Lockerd, R. M.
McChesney, T. D.
McDonald, Marshall
Nicolau, N. G.
Rahman, Arifur
Rekoff, M. G., Jr.
Ryan, R. M.
Slough, J. M.
Spence, D. W.
Winslow, J. D., Jr.
Zerby, J. C.

Kansas City

Clarke, R. L.
Fayman, D. L.
Gareis, G. E.
Halijak, C. A.
Hall, J. B. B.
Ho, J. P. L.
Johns, W. L.
Johnson, R. E.
Miller, C. V.
Nelson, J. W.
Reynolds, C. H.
Rhodes, M. H.
Simonds, R. L.
Stout, H. L.
Webb, G. W.
Wilcox, J. V.

Little Rock

Fox, D. N.
Lewis, H. Z.
Papaleonardos, Dimitris
Sizemore, G. M.
Wilson, J. L.

Lubbock

Crowder, L. F.
Griffith, P. G.
Markwardt, A. J.

New Orleans

Cronvich, J. A.

Oklahoma City

Challenner, A. P.
Glenn, L. R.
Ledbetter, R. P.
Mitchell, W. S.
Murta, E. J., Jr.
Watson, J. K.

St. Louis

Arndt, R. L.
Beuc, Roman, Jr.
Boman, W. T., Jr.
Bowers, J. C.
Bundschuh, W. A.
Crow, R. K.
Deuschle, R. C.
Diesel, J. W.
Dunn, L. R.
Esterson, G. L.
Fiedler, G. J.
Flake, R. H.
Haselwood, Scotty
Hieken, M. H.
Homoky, L. E.
Largo, G. V.
Lauher, V. A.
Lindenberg, E. C.
Malsbary, J. S.
Martin, W. G.
McAninch, C. H.
Min, H. S.
Mohrman, R. F.
Mundel, E. F.
Reed, D. L.
Rosenkoetter, E. A.
Sayer, J. D.
Scherz, C. J.
Schmidt, E. C., Jr.
Sheehan, J. S.
Shockley, L. W.
Sudfeld, C. C.
Taylor, Willie
Thomson, J. R.
Tucker, M. F.
Twombly, J. W., Jr.
Vieth, R. V.
Wagner, F. R.
Waters, J. I.
Zaborszky, John

San Antonio-Austin

Bostick, F. X.
Hoffman, A. A. J.
Lowenberg, E. C.
Phillips, J. P.
Saunders, M. M.
Simpson, S. H., Jr.
Thomas, M. J., Jr.

Shreveport

Gordon, Edward
Long, F. V.
Woodard, T. A.

Tulsa

Borland, David
Brashear, R. T.
Day, C. E.
Freeman, L. R.
George, L. L.
Goldwyn, R. M.
Hopkinson E. C.
Keene, L. C.
Labarthe, L. C.
Patrick, J. R.
Piersall, B. K.
Piety, R. G.
Silverman, Daniel
Tartar, John
Wiik, Erik

Wichita

Damaskos, N. J.
Mantey, J. P.
O'Loughlin, J. B.

REGION 7

Alamogordo-Holloman

Guenthner, F. K.
Koellner, K. H.
Liston, D. H.
Peake, E. C.

Albuquerque-Los Alamos

Bennett, H. A.
Demuth, H. B.
Fisher, H. V., Jr.
Green, M. C.
Hayre, H. S.
Hu, C. T.
Joppa, R. M.
Koschmann, A. H.
Linsenmayer, G. R.
Milich, C. P.
Mohler, R. R.
Pace, T. L.
Powell, R. L.
Ray, H. K.
Singer, Sidney
Soong, An-Hwa
Spokas, F. J.
Todd, J. L., Jr.
Tompkins, H. E.
Warner, B. D.
Whaley, E. W.

Anchorage

Gauss, E. J.

China Lake

Brandt, R. D.
Creusere, M. C.
Davis, J. E.
Dull, M. J.
Foster, F. N.
Glatt, Benjamin
Kim, P. K.
Reuben, E. M.
Schneider, N. J.
Taylor, J. C.

Fort Huachuca

Frese, R. E.
Lamb, J. J.

Los Angeles

Abernathy, O. R., Jr.
Abzug, M. J.
Adler, J. A.
Akin, P. A.
Albrecht, Albert
Albright, T. B.
Allder, J. R.
Allen, G. F.
Allen, R. J.
Alper, S. M.
Alpine, P. E.
Alvine, C. E.
Ambrose, J. R.
Anderosky, Eugene
Anderson, F. A.
Angle, S. L.
Antul, J. J.
Aoki, Masanao
Apa, F. E.
Arnold, J. R.
Aroyan, G. F.
Arsenault, W. R.
Aseltine, J. A.
Aulenbrock, J. C.
Auletti, R. L.
Avrech, Norman
Axelband, E. I.
Baerg, H. R.
Bainbridge, B. O.
Baker, C. R.
Baker, D. L.
Balducci, J. D.
Balluff, R. L.
Barker, A. C.
Barlett, F. R.
Barnett, Leon
Bartholomew, H. R.
Beecher, D. E.
Beers, K. H.
Bejack, Benton
Bekey, G. A.
Bekey, Ivan
Bement, W. A.
Bemis, R. C.
Bennett, W. E.
Benning, F. N.
Bernstein, Theodore
Bible, R. E.
Bills, G. W.
Bird, R. M.
Blair, L. L.
Bloom, Vitaly
Blount, G. H.
Blum, L. M.
Bold, N. T.
Bolman, R. G.
Bolster, C. A.
Bonney, R. B.
Booton, R. C.
Borgeson, P. W.
Bourbeau, F. J.
Bouse, G. B.
Bower, J. L.
Boxer, Rubin
Boyd, A. F.
Boykin, T. R., Jr.
Bradburn, J. R.
Braddock, R. C.
Brandt, Ralph
Braun, E. L.
Brewer, H. E.
Brigdon, J. K.
Broadwell, W. B.
Brown, L. E.
Buesseler, M. L.
Buland, R. N.
Bunge, W. R.
Burns, S. S.
Burnsweig, Joseph, Jr.
Caine, S. H.
Caler, J. W.
Callot, Sherman
Campbell, Graham
Cannon, C. D.
Capps, J. W.
Carlson, C. O.
Carpenter, D. D.
Carrier, G. T.
Cassidy, L. D.
Chamberlain, A. T.
Chamorro, R. D.
Chandaket, Prapat
Chang, Bansum
Chang, Y. N.
Chapman, C. W.
Chapsky, Jacob
Chase, H. E.
Cherniack, A. E.
Chestnut, M. G.
Christensen, A. V.
Christner, W. G.
Chrystie, Henry
Clark, P. A.
Cluck, H. D.
Cochràn, E. D.
Cohill, G. L.
Collins, W. C.
Collmer, R. C.
Cooper, A. F.
Cooper, D. L.
Cornell, J. R.
Cosgrave, S. J.
Cowart, B. E.
Crews, J. B.
Cribbs, W. J.
Cromleigh, R. G.
Crooke, V. E.
Crowe, J. W.
Cutting, Elliott
Czah, A. J.
Dao, T. T.
Davidson, R. G.
Davis, D. C.
Davis, F. W.
Davis, H. G., Jr.
Davis, J. C.
Davis, T. F.
Davisson, J. R.
Deaux, F. J.
Deen, J. R.
Dempsey, A. D.
Deuser, D. A.
Dickerson, D. H.
Diemer, F. P.
Dinning, J. R.
Distaso, J. R.
Dodd, G. M.
Dolce, S. L.
Doty, R. L.
Doyle, P. A., Jr.
Drake, R. L.
Driver, D. L., Jr.
Drucker, A. N.
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Durand, Tulvio, S.
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Edelsohn, C. R.
Eggeman, D. J.
Egger, Alexander
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Eikelman, J. A., Jr.
Elgin, J. H.
Ellis, Alvin
Ellis, D. O.
Engel, H. L.
Ennis, F. L.
Eno, R. F.
Epstein, H. C.
Erway, D. D.
Estrin, Gerald
Fiorentino, J. S.
Fischer, P. F.
Fish, K. L.
Fisher, R. C.
Fisher, W. E.
Fitzgerald, G. E.
Fleeman, P. J.
Floyd, G. W.
Flynn, D. T.
Fortier, R. E.
Foster, W. H.
Fox, A. J.
Fox, S. R.
Foxman, Eugene
Frank, C. E.
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Frankos, D. T.
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Frith, L. O.
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Gabler, R. T.
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Gaston, J. L.
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Gee, L. C.
George, R. D.
Gerardi, Francis
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Gillis, D. E.
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Goodwin, J. W.
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Griep, D. J.
Griffith, S. D.
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Guerre, C. L.
Gullatt, S. P., Jr.
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Gustin, J. T.
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Hailey, R. D.
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Hall, C. R.
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Harriman, T. J.
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Holmen, R. E.
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Izuel, A. G.
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Kitabayashi, Tamotsu
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Schultz, P. R.
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Schweitzer, B. P.
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Seitz, F. R.
Seltzer, L. J.
Sensiper, Samuel
Serna, R. C.
Shapiro, Arthur
Shearer, G. W.
Shein, Saul
Shell, Stanley
Shelley, R. G.
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Shuler, M. H.
Shumate, M. C.
Shutt, S. G.
Siegel, J. C.
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Simpson, Bob L.
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Small, R. H.
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Smith, J. D.
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Smith, R. E.
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Snapp, K. M.
Sofen, I. A.
Sohler, J. F.
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Spadaro, F. G.
Speen, G. B.
Spring, L. K.
Staub, D. W.
Staudhammer, John
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Stein, Leonard
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Sterrett, J. P.
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Stout, T. M.
Stribling, R. E.
Stringer, R. B.
Stubberud, A. R.
Sturm, T. F.
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Summerfield, A. R.
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Swanson, A. L.
Swiger, J. M.
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Szirmay, S. Z.
Takahashi, Koyoshi
Tallman, C. R.
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Taylor, C. W.
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Tracey, B. P.
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Trumbo, D. E.
Turn, Rein
Udry, J. J.
Valette, H. C.
Valery, N. A.
Van Curen, Verlyn
Van Wechel, R. J.
Vega, C. J.
Verano, Rank
Vincent, G. D.
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Walker, E. S.
Walp, R. M.
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Walters, L. G.
Ware, P. H.
Wasney, Albert
Watanabe, A. S.
Waterman, H. B.
Watkins, E. L.
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Wedel, J. J., Jr.
Weiss, Mitchell
Welken, R. L.
Whelan, D. E.
White, K. E.
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Whitford, R. K.
Whiting, Lee
Wichmann, T. F.
Wilderman, L. F.
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Williams, A. S.
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Wilson, J. A.
Wilts, C. H.
Winters, M. N.
Winton, H. L.
Witt, A. N.
Wolf, M. B.
Wolman, L. L.
Wright, P. B.
Wunderlich, F. J.
Wyatt, W. C.
Wylie, Lluis
Yamamoto, T. G.
Yee, Gene
Yonemoto, Noboru
Young, G. O.
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Zoll, D. J.
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Chapin, D. W.
Cluff, L. W.
Cummings, K. C.
Elsner, R. W.
Gaines, W. M.
Glass, T. J.
Hodson, R. B.
Ittenbach, L. J., Jr.
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Matava, S. J.
Montgomery, E. B.
Osder, S. S.
Peterson, R. K.
Richman, M. A.
Ross, J. M.
Russell, W. F.
Sanneman, R. W.
Schwarz, F. W.
Scott, D. E.
Severns, R. A.
Shipley, P. P.
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Tom, Donald
Wallace, R. A.
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Hedrick, L. C.
Kelly, R. L.
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Peterson, D. E.
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Evans, D. S.
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Gottwald, C. H.
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McLauchlan, J. M
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Prager, R. H.
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Seabaugh, D. A.
Shechet, M. L.
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Waddell, Bill L.
Wade, Ernest
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Wexler, H. T.
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Kunnes, Herbert
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Lee, H. Q.
Landaris, G. G.
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Linden, D. A.
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Woolsey, W. O.
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Wye, R. E.
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Godo, Einar
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Louie, C. H.
Lurz, R. F.
Manetsch, T. J.
Matthews, R. A.
McElhoe, B. A.
McGonagle, L. E.
McPherson, E. L.
Minkema, J. H.
Nelson, J. K.
Noges, Endrik
Noland, L. J.
Page, Ronald
Retka, J. A.
Ruiz, A. L.
Sastry, P. V.
Skahill, B. J., Jr.
Smith, K. C.
Spencer, K. J.
Stiede, W. L.
Tenning, C. B.
Tubb, D. G.
Upright, R. D.
Vermilion, E. E.
Wayment, M. D.
Wehlander, E. O.
Wheeler, P. C.
Zeutschel, M. F.

Tucson

Bahin, L. J., Jr.
Balk, Sheldon
Clark, H. L.
Dloughy, F. W.
Fish, W. Y.
Giles, D. L.
Glaesser, Ewald
Hausenbauer, C. R.
Kinnie, I. G.
Lynch, J. A., Jr.
Martin, L. C.
Maybock, R. L.
Michalk, E. R.
Morgan, M. J.
Peterson, G. R.
Pierce, E. P., Jr.
Pilling, L. H.
Sage, A. P., Jr.
Sakrison, D. J.
Terrazas, Carlos
Wozny, M. J.

REGION 8

Bay of Quinte

Broughton, M. B.
Dickout, H. C.
MacKelvie, J. S.
Revill, A. D.
Young, J. A. I.

Hamilton

Berger, Hellmuth
Farkas, E. C.
Kassner, John
Sparrow, James
Watts, T. O.
Waud, E. C.

Kitchener-Waterloo

Myers, B. R.

London

Fletcher, H. R.

Montreal

Bar-Urian, Moshe
Baumans, H. W.
Belanger, P. R.
Birman, Gerhard
Breton, A. Y.
Cadieux, J. P.
Cameron, B. G.
Capper, A. N.
Cox, J. R. G.
Dingwall, R. A.
Gravel, J. J. O.
Guitian, J. E.
Haeberlin, R. O. W.
Hall, C. D.
Hauck, H. G.
Heckman, D. E.
Howarth, B. A.
Jamshedji, J. S.
Kubina, S. J.
Lindsay, R. A.
Lyle, S. M.
MacDonald, A. G.
Maclennan, N. D.
Medweth, W. D.
Miller, W. R.
Newton, K. G. D.
Oxley, A. B.
Reeves, Rene
Schmidt, P. J.
Semple, E. R.
Smolinski, W. J.
Szot, H. J.
Tang, D. D. Y.
Warener, Glenfield
Watson, J. C.
Wood, H. H.
Zames, G. D.

Newfoundland

Alway, V. J.

Northern Alberta

Ropchan, D.
Szojka, F. G.

Ottawa

Banga, Mike
Bechthold, G. H.
Borlase, W. N.
Brunton, H. W.
Chouinard, P. H.
Chrzanowski, J. T.
Clemence, C. R.
Cole, W. A.
Cowper, George
Dietiker, Walter
Funke, E. R. R.
Glinski, George
Goulding, F. S.
Levy, M. M.
MacLulich, D. A.
Norton, J. A.
Pritchard, J. R.

Quebec

Boulet, Lionel
Caron, J. Y.
L'Heureux, L. J.
Newman, P. M.

Regina

Buhr, R. J.
Pylyshyn, Z. W.
Ratz, H. C.
Un, K. K.

Southern Alberta

Cole, A. M.
Johnston, C. W.
Umbach, E. A.

Toronto

Andrau, W. J.
Baldwin, J. H.
Blois, W. G.
Byers, H. G.
Campbell, J. H.
Carew, S. J. H.
Cowan, D. D.
Eakins, J. R.
Fisher, D. E.
Freedman, M. D.
Hackbusch, R. A.
Kelk, G. F.
Martin, J. D. H.
McClean, R. K.
Millen, T. I.
Otsuki, J. S.
Penrose, R. M.
Pile, J. H.
Rowe I. H.
Ryva, G. J.
Slapsys, A. P.
Stein, Alfred
Stephenson, W. R.
Wong, R. D. C.

Vancouver

Bohn, E. V.
Graham, D. A.
Hartley, D. C.
Kasvand, Tonis
Mohammed, Auyuab
Moore, A. D.
Pettersen, G. W. E.
Pirart, M. A.
Stone, R. N.

Winnipeg

Ewashko, T. A.
Hill, J. D.
Ivey, K. A.
Johnson, R. A.
Musick, M. P.

OVERSEAS MILITARY

Borlik, R. E.
Brooks, H. B.
Chandler, D. P.
Head, G. M.
Knausenberger, G. E.
Nothman, M. H.
Pendleton, F. L.
Reich, R. F.
Spafford, L. J.
Thayer, W. D.

FOREIGN SECTIONS

Benelux

Bijl, Aart
Cogneaux, M.
Cosaert, R. L.
Ensing, Lukas
Groenenboom, Maarten
Heeroma, H. H.
Hoffman, J. A. J. L.
Janssen, J. M. L.
Jespers, Paul
Murphy, Bernard
Nanni, Bino
Rubio, J. C.
Van Egmond, A. J. L.
Van Nauta Lemke, H. R.
Varekamp, J. F.

Buenos Aires

Flint-Halpern, Manuel
Pinasco, S. F.
Theis, Rodolfo

Egypt

El Demerdache, A. R. M.
El-Sabbagh, H. H.
Kamal, A. A.
Mikhail, S. L

India

Joshi, M. G.
Kalani, P. W.
Kamat, D. S.
Mirchandani, I. T.
Mishra, Srinibas
Prabhakar, Alladi
Shouri, J. C.
Sinha, N. K.

Israel

Baneth, Michael
Bonenn, Z.
Brandman, Z. I. A.
Herzberg, E. N.
Joselevich, Meir
Kamil, Tsvi
Kass, Sholom
Kliger, I. A.
Mahler, Yona
Mass, Jonathan
Merchav, S. J.
Mor, Rephael
Mydansky, Dan
Pikarsky, Joel
Raviv, Abraham
Samuel, Sergiu
Schoen, T. E.
Schoor-Kon, J. J.
Seider, Mexahem
Shamir, Jedidiah
Weinman, Joseph
Weislitzer, Josef

Italy

Biondi, Emanuele
Bonghi, Mario
D'Auria, Giovanni
De Dominicis, C. M.
Derossi, A. D.
De Vito, G. R.
Ercoli, Paolo
Floriani, Virgilio
Foddis, Guiseppe
Giani, A. S.
Landesman, S. O.
Lombardi, Mario
Maione, Guido
Marchesini, Giovanni
Missio, D. V.
Morganti, Ignazio
Palandri, G. L.
Pinolini, F.
Prette, Mario
Quaglia, Giangiacomo
Rebaudengo, Sergio
Rinaldi, Massimo
Silvestri, Silvestro
Tchou, Mario
Tiberio, Ugo
Schiaffino, Paolo
Vergani, Angelo

Rio de Janiero

Maia, G. N. S.

Tokyo

Aoi, Saburo
Azakami, Takashi
Ezoe, Hirohiko
Fujisawa, Masatoshi
Fukata, Msao
Gohara, Sawako
Harada, Naofumi
Harashima, Osamu
Hata, Shiro
Hayashi, Isuo
Ibuka, Masaru

Imahata, Kenroku
Imai, Haruzo
Ishikawa, Denji
Iwakata, Hideo
Kadokura, Toshio
Katayama, Aisuke
Kawano, Kouji
Kimura, Rokuro
Kinase, Matsunaga
Koichibara, Tadashi
Kojima, Yoshiaki
Komota, Takuya
Konomi, Mitsugu
Koomura, Shuhei
Kosako, Hideo
Kumagai, Mituru
Masada, Eisuke
Matsuyuki, Toshitada
Mikami, Tetsuzo
Minozuma, Fumio
Miyakoshi, Kazuo
Mizuma, Masaichiro
Morimiya, Takaji
Morita, Kazuyoshi
Morita, Masasuke
Murono, Nobuyoshi
Nagami, Kohichi
Nagata, Akira
Nakagami, Minoru
Nakajima, Masaya
Nakahara, Fujio
Nishino, Osamu
Niwa, Yasujiro
Nojima, Susumu
Ogawa, Toru
Okada, Minoru
Okamura, Sogo
Okura, Tsunehiko
Ozaki, Hiroshi
Saito, Yukio
Sawada, Ryoka
Sawada, Toshio
Kohno, Shishu
Kobayashi, Kengo
Kitsuregawa, Takashi
Seki, Hideo
Shinkawa, Hiroshi
Suzuki, Michiya
Tanaka, Yonfji
Taniguchi, Fusao
Terao, Mitsuru
Togino, Kazuto
Tomono, Masami
Tomota, Miyaji
Toyoda, Junichi
Tsumura, Toshihiro
Watanabe, Akira
Yamazaki, Takashi
Yonezawa, Shigeru

FOREIGN COUNTRIES

Arabia

Cook, G. J.

Argentina

Duran, C. A.

Australia

Aitchison, R. E.
Brodribb, M. I.
Coombs, J. S.
D'Assumpcoa, H. A.
Davies, R. J. C.
Ellesworth, George
Honnor, W. W.
Soetratma, Dzaka

Austria

Zemanek, Heinz

Brazil

Benchimol, Augusto
Boffi, L. V.
De Barros Barreto, L. A.
De Mesquita, A. L. P.
Girardelli, L. R. L.
Kempenich, Geraldo
Launberg, E. T.
Quintaes, S. B.
Schwager, C. A. H.
Waeny, J. C. C.

Czechoslovakia

Djadkov, Sergej
Horna, O. A.
Sedmidubsky, Zdenek

Denmark

Andersen, K. T.
Hansen, G. K. F.
Hansen, Kjeld
Rasmussen, Jorgen
Svensson, Flemming
Thirup, Gunnar

England

Barlow, Derek
Barlow, H. E.
Bonenn, Zeev
Booth, P. C.
Bowden, B. V.
Bunzl, T. F.
Cattanes, Edward
Clewes, A. B.
Coffee, R. A.
Cullen, A. L.
Dalgleish, J. W.
Davies, D. L.
Dawe, F. W.
Espley, D. C.
Fischbacher, R. E.
Harris, K. E.
Hindle, Harold
Jackson, W. R.
Jackson, Willis
James, J. R.
Knowles, J. B.
Laverick, Elizabeth
Lemay, L. P.
Lubbock, J. K.
Marzolini, R. G. A.
Minton, Merton
Montgomerie, G. A.
Pearson, J. D.
Robertson, A. A.
Rose, S. P.
Savage, W. B.
Watts, D. G.
Wood, James

Finland

Luoto, U. A.

France

Angot, A. M.
Baron, Jean
Barret, J. P.
Berline, S. D.
Bodez, P. M.
Brachet, Claude
Carrus, P. A.
Deschamps, J. D.
Deschamps, R. A.
Ferrier, P. A.
Ghertman, Jean
Girerd, J. L. M.
Jarleton, Paul
Klein, G. J. P.
Labin, Emile
Lehmann, G. J.
Lermoyez, M. J.
Lestel, J. H.
Oudard, Albert
Prache, P. M.
Raymond, F. H.
Sokoloff, B. A.
Sold, Uzi
Victor, C. P. A.

Germany

Borgnis, F. E.
Busch, C. W.
Conrad, D. A.
Effertz, F. H.
Klamka, Norbert
Peters, J. F.
Tsui, F. F.
Walther, Alwin

Greece

Icihhakis, M. A.

Hong Kong

Tsang, Shing-Fat

Iran

Nuban, Ebrahim

Ireland

Allen, T. P.
Bond, A. D.
Kellway, B. J.

Jugoslavia

Bremasak, Franc

Korea

Choi, Bang-Jin

Mexico

Aguilora, Augusto
Alvarez, J. B.
Chavez, Napoleon
Diaz, J. S.
Higuera-Mota, H. R.
Rodriques, Eugenio
Whidden, G. H.

Nicaragua

Guerrero, M. A.

Norway

Haralsen, H. P.
Lied, Finn
Mack, Erik

Okinawa

Onaga, Kenji

Rumania

Toma, V. I.

Scotland

Mascall, A. L.

Singapore

Khor, T. H. M.
Lye-Hon, Ho

South Africa

Phillips, W. E.
Zawels, Jakob

Spain

Fernando, G. A.
Gomez, J. H.
Gonzalez, J. C.

Sweden

Andersson, K. N.
Anjou, L. A.
Arosenius, L. T. H.
Aslund, N. R. D.
Carlsson, L. G.
Ekelof, Stig
Fagerlind, S. G.
Faxo, Lennart
Gyllenkrok, Thure-Gabriel
Hainer, P. A.
Hellgren, G. P.
Heyman, B. C.
Josephson, B. A. S.
Katzeff, Kurt
Knutsson, Civilingenjor
Laurent, Torbern
Lofgren, E. O.
Nilsson, B. N. A.
Olving, Sven
Perers, O. F.
Persson, N. I. E.
Richter, H. F.
Roll, Anders
Sivers, C. H. V.
Wikland, T. E.

Switzerland

Angehrn, T. W.
Bachmann, A. E.
Baumgartner, R. H.
Braun, A. F.
Diggelmann, Hans
Gayer, J. H.
Roch, A. A.
Shah, R. R.
Simpson, B. H.
Strohschneider, Walter
Thalmann, Victor
Villars, C. P.

Thailand

Pricha, S. C.

U.S.S.R.

Goviadinov, Vladimir
Kostanjanz, B. A.

Venezuela

Arreaza, R. G.
Huerta, A. F.
Stephens, T. L.

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